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**NIEDERAUER, George Frederick, 1942-  
NEUTRON KINETICS BASED ON THE EQUATION OF  
TELEGRAPHY.**

**Iowa State University, Ph.D., 1967  
Engineering, nuclear**

**University Microfilms, Inc., Ann Arbor, Michigan**

NEUTRON KINETICS BASED ON THE EQUATION OF TELEGRAPHY

by

George Frederick Niederauer

A Dissertation Submitted to the  
Graduate Faculty in Partial Fulfillment of  
The Requirements for the Degree of  
DOCTOR OF PHILOSOPHY

Major Subject: Nuclear Engineering

Approved:

Signature was redacted for privacy.

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1967

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DEDICATION

To Mom

Dad

Rose

Mark

Mike

## I. INTRODUCTION

Much of the theory of neutron kinetics has been based on the diffusion approximation for neutron distributions. On the whole this approach has been adequate in explaining the results of many experiments and useful in the stability studies of reactors. An exact analysis of reactor dynamics requires the use of the time-dependent Boltzmann transport equation, but this equation is too unwieldy for many experimental and engineering applications.

For one-velocity neutrons the spherical harmonics expansion of the angular dependence of the neutron flux provides a simple and useful form for reactor analysis. For time-independent systems the simplest of the spherical harmonic expansions, the  $P_1$  approximation, is equivalent to the diffusion approximation (see, for example, the deduction by Megrebian and Holmes (1)). The time-dependent  $P_1$  equations, however, are not equivalent to the time-dependent diffusion equation, as will be shown in chapter III.

Descriptions of diffusion phenomena have the essential feature that all disturbances propagate with "infinite" velocity; that is, any disturbance at any point in the system is felt everywhere instantly. On the other hand, the two equations of the  $P_1$  approximation can be combined into a single equation, the telegrapher's (telegraphist's) equation, which describes physical phenomena exhibiting both wavelike characteristics and residual disturbance effects. The wavelike behavior is described by the second-order time derivative, and the residual disturbance effect, by the first-order time derivative. Wave effects propagate with a finite velocity; consequently, the introduction of the second-order term is necessary for the description of time-wise variations in the neutron population.

However important the second derivative in time is conceptually, the general adequacy of diffusion theory indicates that it is relatively unimportant in reactor calculations. It is important to know the conditions for which diffusion theory holds. Investigation of the telegrapher's equation should help to determine some of the limits on the adequacy of diffusion theory.

Although many authors (1-4) have commented on the telegrapher's equation, little has been done with it in connection with nuclear reactor theory until recently. Interest in wavelike behavior of neutrons began when Weinberg and Schweinler published their classic paper (5) in 1948. The experimental work of Raievski and Horowitz (6), reported at the 1955 Geneva Conference, and of Uhrig (7) in 1959, and the theoretical work of Perez and Uhrig (8) in 1963 renewed the interest in neutron waves, and much work has been reported since then.

The growing interest in neutron wave propagation led to Pasquantonio's general approach (9) to the telegrapher's equation in 1964 and Mortensen and Smith's (10) and Uhrig's (11) studies of wave propagation in 1965 using the  $P_1$  approximation. These studies involved rectangular coordinate systems. Hale (12) solved the telegrapher's equation in spherical geometry for a pulsed point source.

The general solution to the telegrapher's equation, for arbitrary constant coefficients and as applied to transmission lines (for example, telegraph lines--from which came the name of the equation) was solved in 1927 by Webster (13) for general initial conditions and an infinite system. He noted that Lord Kelvin treated this equation in 1855 for a submarine cable. Then in 1956 Doetsch (14) published a solution of this type for a semi-

infinite system (a special case was handled by Doetsch (15) in 1937). Re-defining the constants and functions, Doetsch's 1956 solution can be identified as the solution for the telegrapher's equation for a semi-infinite, homogeneous nuclear system, as presented by Pasquantonio (9).

The general solution to the telegrapher's equation for zero initial conditions in reactor theory is different from that for transmission lines since the boundary conditions differ. Pasquantonio chose a pseudo-diffusion boundary condition (to be discussed more in chapter III) to solve for this general solution. In chapter IV of the present work a solution is presented for a boundary condition which is consistent with the  $P_1$  equations.

Closely related to the wave propagation technique is the pulse propagation technique. The former considers single frequencies, one at a time, and the latter, a certain fundamental frequency and its harmonics. If one investigates a point oscillating absorber in a reactor, the analysis of Weinberg and Schweinler provides the space-time solution using diffusion theory. At high frequencies the oscillating absorber generates spherical and higher-order waves; at low frequencies the response may be described by the spatially independent reactor transfer function. The reactor transfer function, that is, the spatially independent transfer function for the linearized one-velocity model, may be determined by the frequency response to uniformly distributed, oscillating absorber. Thus the work of Mortensen and Smith and of Uhrig on wave phenomena can be extended to considerations of neutron pulse propagation and reactor transfer functions. These subjects are explored in chapters V and VI, respectively.

## II. OBJECTIVES AND SCOPE

The objectives of this work are two-fold. The first is to correct Pasquantonio's work by presenting a consistent approach to the solution of the telegrapher's equation for the case of an arbitrary source at a boundary of one-dimensional media having zero initial conditions. The second is to compare the telegrapher's equation with the diffusion equation, thereby determining some of the limitations on diffusion theory.

In the introduction it was mentioned that comparisons have been made between the telegrapher's and diffusion equations for neutron wave propagation. Neutron pulse propagation and reactor transfer functions are closely related to neutron wave propagation and will be examined here, in chapters V and VI respectively. The two propagation techniques are specific cases of the general problem, solved in chapter IV, of an arbitrary source located at a boundary. The case of a sinusoidal source located at a boundary is examined in chapter IV also. The problems thus far mentioned are spatially dependent, except for the reactor transfer function. An investigation of spatially independent kinetics of the telegrapher's equation is presented in chapter VI. The point reactor kinetics equations are derived from the telegrapher's equation. This derivation leads into a derivation of the reactor transfer function, which is the response of a reactor to an internal, oscillating source. These investigations are carried out for two particular geometries: a semi-infinite medium and an infinite slab.

### III. FORMULATION OF EQUATIONS

#### A. The $P_1$ Approximation

The most accurate description of the neutron population in a system is given by the Boltzmann transport equation, which is a complex integro-differential equation. However, for many problems adequate results are obtained from elementary transport theory or diffusion theory if the following conditions hold (4): i. absence of inelastic scattering, ii. spherically symmetric scattering in the center of mass coordinate system, and iii. lack of chemical binding effects. Only monoenergetic neutrons in isotropic and homogeneous media are considered here. The appropriate form of the transport equation for the neutron flux  $\phi(\vec{r}, \vec{\Lambda}, t)$  under these conditions is

$$\begin{aligned} & \frac{1}{v} \frac{\partial}{\partial t} \phi(\vec{r}, \vec{\Lambda}, t) + \vec{\Lambda} \cdot \nabla \phi(\vec{r}, \vec{\Lambda}, t) + \sum_t \phi(\vec{r}, \vec{\Lambda}, t) \\ &= S(\vec{r}, \vec{\Lambda}, t) + \int_{\vec{\Lambda}'} \sum_s \phi(\vec{r}, \vec{\Lambda}', t) f(\vec{\Lambda}; \vec{\Lambda}') d\vec{\Lambda}' . \end{aligned} \quad (1)$$

where  $v$  is the neutron speed,  $\sum_t$  and  $\sum_s$  are the total and scattering cross sections,  $\vec{\Lambda}'$  is the direction of motion,  $f(\vec{\Lambda}; \vec{\Lambda}')$  is the scattering function relating the incident direction  $\vec{\Lambda}'$  to the scattered direction  $\vec{\Lambda}$ , and  $S(\vec{r}, \vec{\Lambda}, t)$  is the source. The solution of this equation is made more tractable by expanding it into spherical harmonics.

For a semi-infinite medium or infinite slab the spatial dependence of the flux and source conditions is reduced to a single space variable  $x$ . As a consequence the system must have azimuthal symmetry, which reduces the direction variable to a single variable  $\theta$ , the angle between the direction of neutron motion and the  $x$  axis. It has been found convenient to express the transport equation in terms of the cosine,  $\mu$ , of the angle  $\theta$  rather

than  $\theta$  itself. The directional flux is expanded in terms of Legendre polynomials by

$$\phi(x, \mu, t) \equiv \sum_{n=0}^{\infty} \frac{2n+1}{2} \phi_n(x, t) P_n(\mu) \quad (2)$$

where

$$\phi_n(x, t) = \int_{-1}^1 \phi(x, \mu, t) P_n(\mu) d\mu. \quad (3)$$

The first two terms of the directional flux have physical significance in the definitions of neutron flux and net current in diffusion theory. From equation 3

$$\phi_0(x, t) = \int_{-1}^1 \phi(x, \mu, t) P_0(\mu) d\mu = \int_{-1}^1 \phi(x, \mu, t) d\mu \equiv \phi(x, t) \quad (4)$$

$$\phi_1(x, t) = \int_{-1}^1 \phi(x, \mu, t) P_1(\mu) d\mu = \int_{-1}^1 \phi(x, \mu, t) \mu d\mu \equiv J(x, t). \quad (5)$$

Weighting the directional flux with its cosine to the  $x$  axis and integrating over the entire range of  $\mu$  gives the net flow in the  $x$  direction, which is called the net current along that coordinate. Replacing  $\phi$  by  $S$  in equations 2 and 3 gives the expansion for the source.

For isotropic media

$$f(\vec{\Omega}; \vec{\Omega}') = \frac{1}{2\pi} \eta(\mu_0; \vec{\Omega}') \quad (6)$$

where

$$\mu_0 \equiv \cos \theta_0 = \vec{\Omega} \cdot \vec{\Omega}'. \quad (7)$$

Thus the scattering is described in terms of the cosine of the scattering angle. The frequency function  $\eta$  is also expanded into terms of the Le-

Legendre polynomials, i.e.,

$$\eta(\mu_0; \vec{n}') \equiv \sum_{n=0}^{\infty} \frac{2n+1}{2} \eta_n P_n(\mu_0) \quad (8)$$

where

$$\eta_n = \int_{-1}^1 \eta(\mu_0; \vec{n}') P_n(\mu_0) d\mu_0. \quad (9)$$

Since the scattering frequency integrated over all directions is unity, the first coefficient is

$$\eta_0 = \int_{-1}^1 \eta(\mu_0; \vec{n}') P_0(\mu_0) d\mu_0 = \int_{-1}^1 \eta(\mu_0; \vec{n}') d\mu_0 \equiv 1 \quad (10)$$

and it follows that

$$\eta_1 = \int_{-1}^1 \eta(\mu_0; \vec{n}') P_1(\mu_0) d\mu_0 = \int_{-1}^1 \eta(\mu_0; \vec{n}') \mu_0 d\mu_0 \equiv \bar{\mu}_0 \quad (11)$$

which is the average value of the cosine of the scattering angle  $\theta_0$ .

The derivation of the  $P_n$  equations from the rigorous transport equation is well-known and may be found in several texts (1, 2, 16). The notation used here follows closely that of Meghreblian and Holmes (1). In a more general method this results in the infinite set of coupled partial differential equations

$$\begin{aligned} -\frac{1}{v} \frac{\partial}{\partial t} \phi_n(x, t) - \sum_t \phi_n(x, t) + \sum_s \eta_n \phi_n(x, t) + S_n(x, t) \\ = \frac{n+1}{2n+1} \frac{\partial}{\partial x} \phi_{n+1}(x, t) + \frac{n}{2n+1} \frac{\partial}{\partial x} \phi_{n-1}(x, t) \end{aligned} \quad (12)$$

$$n = 0, 1, 2, \dots$$

but in the case of an isotropic medium the source is assumed to be isotropic also and a minimum of anisotropy is assumed in the flux. The

first assumption states that  $S_n(x, t) = 0$  for  $n \neq 0$ . The second truncates the set of equations at  $n=1$ ; letting  $n=0, 1$ , the resulting set of equations, called the  $P_1$  approximation, is

$$-\frac{1}{v} \frac{\partial}{\partial t} \phi_0(x, t) - \sum_t \phi_0(x, t) + \sum_s \phi_0(x, t) + S_0(x, t) = \frac{\partial}{\partial x} \phi_0(x, t) \quad (13)$$

$$-\frac{1}{v} \frac{\partial}{\partial t} \phi_1(x, t) - \sum_t \phi_1(x, t) + \sum_s \bar{\mu}_0 \phi_1(x, t) = \frac{1}{3} \frac{\partial}{\partial x} \phi_0(x, t). \quad (14)$$

### B. The Telegrapher's Equation

Taking the partial derivative of equation 13 with respect to time and of equation 14 with respect to space, eliminating the cross partial derivatives between them, and substituting equation 13 for  $(\partial/\partial x) \phi_1(x, t)$  yields the telegrapher's equation for neutron transport:

$$\begin{aligned} & \frac{3D}{v^2} \frac{\partial^2}{\partial t^2} \phi(x, t) + \frac{1}{v} (1 + 3D \sum_a) \frac{\partial}{\partial t} \phi(x, t) \\ &= D \frac{\partial^2}{\partial x^2} \phi(x, t) - \sum_a \phi(x, t) + S(x, t) + \frac{3D}{v} \frac{\partial}{\partial t} S(x, t) \end{aligned} \quad (15)$$

where

$$D \equiv \frac{1}{3(\sum_t - \sum_s \bar{\mu}_0)} = \frac{1}{3 \sum_{tr}}, \quad (16)$$

$S(x, t) = S_0(x, t)$ , and  $\phi_0(x, t) = \phi(x, t)$ . Comparing this with the diffusion equation

$$\frac{1}{v} \frac{\partial}{\partial t} \phi(x, t) = D \frac{\partial^2}{\partial x^2} \phi(x, t) - \sum_a \phi(x, t) + S(x, t) \quad (17)$$

reveals three differences: (a) an additional term involving the second time derivative of the flux, (b) an additional term involving the time derivative of the source, and (c) the alteration of the coefficient of the first derivative of the flux. Weinberg and Wigner (2), Meghrebian and Holmes (1), and Ash (3) consider only time independent sources, thereby neglect-

ing  $(\partial/\partial t) S(x,t)$ . Isbin (4) considers a fission source which hides the distinction in the coefficient of the time derivative of the flux. The nature of the time dependence of the source is more lucid when it is stated explicitly.

The factor  $3D/v$  appears only in those terms in which a difference exists between the telegrapher's and diffusion equations and is the largest common factor of those coefficients. This term may be rewritten as

$$\frac{3D}{v} = \frac{1}{v \sum_{tr}} = \frac{\lambda_{tr}}{v} = \tau_0 \quad (18)$$

which is now defined as the infinite mean neutron transport time, or, simply, the transport time (the finite transport time  $\tau$  will be discussed in chapter VI). Equation 15 becomes

$$\begin{aligned} & \frac{\tau_0}{v} \frac{\partial^2}{\partial t^2} \phi(x,t) + \left( \frac{1}{v} + \tau_0 \sum_a \right) \frac{\partial}{\partial t} \phi(x,t) \\ &= D \frac{\partial^2}{\partial x^2} \phi(x,t) - \sum_a \phi(x,t) + S(x,t) + \tau_0 \frac{\partial}{\partial t} S(x,t). \end{aligned} \quad (19)$$

The physical significance of the transport time is that it describes the time delay between the cause and effect at two distinct points in an actual system. It will be noted that the telegrapher's equation reduces to the diffusion equation when  $\tau_0 = 0$ . Weinberg and Wigner perform the same reduction by taking the limit as the mean free path  $\lambda_{tr}$  and the absorption cross section  $\Sigma_a$  approach zero while the neutron velocity  $v$  approaches infinity, so that the diffusion coefficient  $vD$  and the infinite mean neutron lifetime  $1/v\Sigma_a$  remain finite.

### C. Boundary Conditions

In chapter IV two types of media are considered: a semi-infinite medium with a vacuum boundary at  $x=0$  and an infinite slab with vacuum bound-

ries at  $x=0$  and  $x=h$ . A source is located at the origin. From equation 2 the  $P_1$  approximation for the directional flux is

$$\phi(x, \mu, t) = \frac{1}{2} \phi_0(x, t) + \frac{3}{2} \mu \phi_1(x, t) \quad (20)$$

and from equation 5 the current is

$$\begin{aligned} J(x, t) &= \int_{-1}^1 \phi(x, \mu, t) \mu d\mu = \int_0^1 \phi(x, \mu, t) \mu d\mu - \int_{-1}^0 \phi(x, \mu, t) \mu d\mu \\ &= J_+(x, t) - J_-(x, t). \end{aligned} \quad (21)$$

Hence

$$J_+(x, t) = \frac{1}{4} \phi(x, t) + \frac{1}{2} J(x, t) \quad (22)$$

$$J_-(x, t) = \frac{1}{4} \phi(x, t) - \frac{1}{2} J(x, t). \quad (23)$$

If an isotropic source is located at the origin and the initial conditions are zero, then

$$S(x, t) = \begin{cases} S(t) u(t) & x=0 \\ 0 & x \neq 0 \end{cases} \quad (24)$$

where  $u(t)$  is the unit step function. The boundary condition is now

$$J_+(0, t) = \frac{1}{2} S(t) u(t) \quad (25)$$

Rewriting equation 14 as

$$J(x, t) = -D \frac{\partial}{\partial x} \phi(x, t) - \tau_0 \frac{\partial}{\partial t} J(x, t) \quad (26)$$

(which differs from the diffusion form by the last term) and combining this result with equation 22 yields

$$2J_+(x, t) + 2\tau_0 \frac{\partial}{\partial t} J_+(x, t) = \frac{1}{2} \phi(x, t) - D \frac{\partial^2}{\partial x^2} \phi(x, t) + \frac{\tau_0}{2} \frac{\partial^2}{\partial t^2} \phi(x, t). \quad (27)$$

Hence the boundary condition at  $x=0$  is

$$\frac{1}{2} \phi(0, t) - D \frac{\partial}{\partial x} \phi(0, t) + \frac{\tau_0}{2} \frac{\partial}{\partial t} \phi(0, t) = S(t)u(t) + \tau_0 \frac{d}{dt} S(t)u(t) \quad (28)$$

Pasquantonio (9) considered the problem of the general solution of the telegrapher's equation, but instead of deriving the proper boundary conditions consistent with the  $P_1$  approximation he chose a pseudo-diffusion boundary condition for the source. He extended the diffusion boundary condition by including the derivative of the source, but ignored the time derivative of the flux, which appears in equation 28. Naturally, including this term makes the solution slightly more complicated than that given by Pasquantonio.

At the vacuum boundary  $x=h$  the condition is

$$\phi(h, t) = 0 \quad (29)$$

In the semi-infinite medium

$$\lim_{x \rightarrow \infty} \phi(x, t) = 0 \quad (30)$$

#### D. Initial Conditions

If no initial distribution exists, the initial conditions are

$$\lim_{t \rightarrow 0^+} \phi(x, t) = \lim_{t \rightarrow 0^+} \frac{\partial}{\partial t} \phi(x, t) = 0 \quad \begin{cases} x > 0 & \text{semi-infinite medium} \\ 0 < x < h & \text{infinite slab} \end{cases} \quad (31)$$

For systems with initial distributions the initial conditions are

$$\lim_{t \rightarrow 0^+} \phi(x, t) = \phi(x) \quad \begin{cases} x > 0 & \text{semi-infinite medium} \\ 0 < x < h & \text{infinite slab} \end{cases} \quad (32)$$

$$\lim_{t \rightarrow 0^+} \frac{\partial}{\partial t} \phi(x, t) = \psi(x) \quad \begin{cases} x > 0 & \text{semi-infinite medium} \\ 0 < x < h & \text{infinite slab} \end{cases} \quad (33)$$

where  $\phi(x)$  and  $\psi(x)$  are the initial flux and its initial time derivative.

## IV. GENERAL SOLUTION: ZERO INITIAL CONDITIONS

## A. Laplace Transforms of Solutions

As already indicated in the preceding chapter the telegrapher's equation will be solved for a semi-infinite medium and an infinite slab with zero initial conditions and with an arbitrary time-dependent source located at the origin, which is also a boundary. The telegrapher's equation is now written without the source terms:

$$\frac{\tau_0}{v} \frac{\partial^2}{\partial t^2} \phi(x,t) + \left( \frac{1}{v} + \tau_0 \sum_a \right) \frac{\partial}{\partial t} \phi(x,t) = D \frac{\partial^2}{\partial x^2} \phi(x,t) - \sum_a \phi(x,t). \quad (34)$$

Denote the Laplace transform of  $\phi(x,t)$  by  $\bar{\Phi}(x,s)$  and let

$$\bar{\Phi}''(x,s) = \frac{\partial^2}{\partial x^2} \bar{\Phi}(x,s).$$

With some rearranging the Laplace transform of equation 34 may be written

$$\bar{\Phi}''(x,s) - k^2(s) \bar{\Phi}(x,s) = 0 \quad (35)$$

where

$$k^2(s) = \frac{\tau_0}{vD} s^2 + \left( \frac{1}{vD} + \tau_0 \frac{\sum_a}{D} \right) s + \frac{\sum_a}{D}. \quad (36)$$

The general solution to equation 35 is

$$\bar{\Phi}(x,s) = A(s) e^{-k(s)x} + B(s) e^{k(s)x}. \quad (37)$$

The Laplace transforms of  $J(x,t)$  and  $J_+(x,t)$  are denoted by  $j(x,s)$  and  $j_+(x,s)$ . Laplace transforming equations 26 and 27 yields

$$(1 + \tau_0 s) j(x,s) = - D \bar{\Phi}'(x,s) \quad (38)$$

$$(1 + \tau_0 s) j_+(x,s) = \frac{1}{4} (1 + \tau_0 s) \bar{\Phi}(x,s) - \frac{D}{2} \bar{\Phi}'(x,s). \quad (39)$$

It follows that

$$j(x, s) = \frac{D k(s)}{1 + \tau_0 s} [A(s) e^{-k(s)x} - B(s) e^{k(s)x}] \quad (40)$$

$$\begin{aligned} j_+(x, s) &= \frac{1}{4} \frac{1 + 2 D k(s) + \tau_0 s}{1 + \tau_0 s} A(s) e^{-k(s)x} \\ &\quad + \frac{1}{4} \frac{1 - 2 D k(s) + \tau_0 s}{1 + \tau_0 s} B(s) e^{k(s)x}. \end{aligned} \quad (41)$$

The coefficients  $A(s)$  and  $B(s)$  are determined by the geometry of the system and the appropriate boundary conditions. They will now be determined for two specific geometries.

### 1. Semi-infinite medium

The boundary conditions are given by equation 30 and by either equation 25 or 28. If the Laplace integral of  $\phi(x, t)$  is continuous with respect to  $x$  and  $t$  and is uniformly convergent with respect to  $x$ , the interchange of the limit and integral operators is valid (17). The Laplace transforms of these conditions are

$$\lim_{x \rightarrow \infty} \Phi(x, s) = \lim_{x \rightarrow \infty} \mathcal{L}\{\phi(x, t)\} = \mathcal{L}\left\{\lim_{x \rightarrow \infty} \phi(x, t)\right\} = \mathcal{L}\{0\} = 0 \quad (42)$$

$$j_+(0, s) = \mathcal{L}\{j_+(0, t)\} = \mathcal{L}\left\{\frac{1}{2} S(t) u(t)\right\} = \frac{1}{2} \sigma(s) \quad (43)$$

$$\frac{1}{2}(1 + \tau_0 s) \Phi(x, s) - D \Phi'(x, s) = (1 + \tau_0 s) \sigma(s) \quad (44)$$

where  $\sigma(s) = \mathcal{L}\{S(t) u(t)\}$ . The condition on equation 37 given by equation 42 can be met only if  $B(s) \equiv 0$ . Hence

$$\Phi(x, s) = A(s) e^{-k(s)x} \quad (45)$$

The coefficient  $A(s)$  may be found by applying the boundary condition at  $x=0$ , that is, either condition 43 to 41, which may now be written as

$$j_+(x, s) = \frac{1}{4} \frac{1 + 2Dk(s) + \tau_s s}{1 + \tau_s s} A(s) e^{-k(s)x} \quad (46)$$

or condition 44 to 37. The result is

$$A(s) = 2 \frac{1 + \tau_s s}{1 + 2Dk(s) + \tau_s s} \sigma(s). \quad (47)$$

Thus for the semi-infinite medium

$$\Phi_\infty(x, s) = \frac{2(1 + \tau_s s)}{1 + 2Dk(s) + \tau_s s} \sigma(s) e^{-k(s)x}. \quad (48)$$

$$j_{+\infty}(x, s) = \frac{1}{2} \sigma(s) e^{-k(s)x} \quad (49)$$

$$j_\infty(x, s) = \frac{2Dk(s)}{1 + 2Dk(s) + \tau_s s} \sigma(s) e^{-k(s)x}. \quad (50)$$

## 2. Infinite slab

The boundary conditions are given by equations 29 and either 25 or 28. The Laplace transforms of conditions 25 and 28 are equations 43 and 44; for condition 29 it is

$$\lim_{x \rightarrow h} \Phi(x, s) = 0. \quad (51)$$

Applying condition 43 to equation 41 and 51 to 37 gives the set of algebraic equations

$$\frac{1}{2} \frac{1 + 2Dk(s) + \tau_s s}{1 + \tau_s s} A(s) + \frac{1}{2} \frac{1 - 2Dk(s) + \tau_s s}{1 + \tau_s s} B(s) = \sigma(s) \quad (52)$$

$$e^{-hk(s)} A(s) + e^{hk(s)} B(s) = 0, \quad (53)$$

which is solved by Cramer's rule, giving

$$A(s) = \frac{2(1+\tau_0 s) \sigma(s)}{1+2Dk(s)+\tau_0 s - [1-2Dk(s)+\tau_0 s] e^{-2hk(s)}} \quad (54)$$

$$B(s) = \frac{-2(1+\tau_0 s) \sigma(s) e^{-2hk(s)}}{1+2Dk(s)+\tau_0 s - [1-2Dk(s)+\tau_0 s] e^{-2hk(s)}}. \quad (55)$$

These coefficients are consistent with those of the semi-infinite medium, for

$$\lim_{h \rightarrow \infty} A_s(s) = A_\infty(s)$$

$$\lim_{h \rightarrow \infty} B_s(s) = B_\infty(s) = 0.$$

Thus for the infinite slab

$$\Phi_s(x,s) = \frac{2(1+\tau_0 s) \sigma(s)}{1+2Dk(s)+\tau_0 s} \frac{e^{-k(s)x} - e^{-(2h-x)k(s)}}{1 - \frac{1-2Dk(s)+\tau_0 s}{1+2Dk(s)+\tau_0 s} e^{-2hk(s)}} \quad (56)$$

$$j_{+s}(x,s) = \frac{1}{2} \frac{e^{-k(s)x} - \frac{1-2Dk(s)+\tau_0 s}{1+2Dk(s)+\tau_0 s} e^{-(2h-x)k(s)}}{1 - \frac{1-2Dk(s)+\tau_0 s}{1+2Dk(s)+\tau_0 s} e^{-2hk(s)}} \sigma(s) \quad (57)$$

$$j_s(x,s) = \frac{2Dk(s)}{1+2Dk(s)+\tau_0 s} \frac{e^{-k(s)x} + e^{-(2h-x)k(s)}}{1 - \frac{1-2Dk(s)+\tau_0 s}{1+2Dk(s)+\tau_0 s} e^{-2hk(s)}} \sigma(s) \quad (58)$$

Since equations 48, 49, 50, 56, 57, and 58 are factorable by  $\sigma(s)$ , it is sufficient to determine the inverse Laplace transforms of

$$f_{\infty}(x,s) = \frac{2(1+\tau_0 s)}{1+2Dk(s)+\tau_0 s} e^{-k(s)x} \quad (59)$$

$$g_{\infty}(x,s) = \frac{1}{2} e^{-k(s)x} \quad \left. \right\} \text{semi-infinite medium} \quad (60)$$

$$h_{\infty}(x,s) = \frac{2Dk(s)}{1+2Dk(s)+\tau_0 s} e^{-k(s)x} \quad (61)$$

$$f_s(x,s) = \frac{2(1+\tau_0 s)}{1+2Dk(s)+\tau_0 s} \frac{e^{-k(s)x} - e^{-(2h-x)k(s)}}{1 - \frac{1-2Dk(s)+\tau_0 s}{1+2Dk(s)+\tau_0 s} e^{-2hk(s)}} \quad (62)$$

$$g_s(x,s) = \frac{1}{2} \frac{e^{-k(s)x} - \frac{1-2Dk(s)+\tau_0 s}{1+2Dk(s)+\tau_0 s} e^{-(2h-x)k(s)}}{1 - \frac{1-2Dk(s)+\tau_0 s}{1+2Dk(s)+\tau_0 s} e^{-2hk(s)}} \quad \left. \right\} \text{infinite slab} \quad (63)$$

$$h_s(x,s) = \frac{2Dk(s)}{1+2Dk(s)+\tau_0 s} \frac{e^{-k(s)x} + e^{-(2h-x)k(s)}}{1 - \frac{1-2Dk(s)+\tau_0 s}{1+2Dk(s)+\tau_0 s} e^{-2hk(s)}} \quad (64)$$

to find  $\phi(x,t)$ ,  $J_+(x,t)$ , and  $J(x,t)$ . The difference between these equations and those found by Pasquantonio (9) is the  $\tau_0 s$  term in the expressions  $1 \pm 2Dk(s) + \tau_0 s$ . The appearance of this term is due to the presence of the time derivative of the flux in the definition of  $J_+(x,t)$ .

If the inverse Laplace transforms of  $f(x,s)$ ,  $g(x,s)$ , and  $h(x,s)$  are  $F(x,t)$ ,  $G(x,t)$ , and  $H(x,t)$ , then the convolution theorem provides the results

$$\phi(x,t) = F(x,t) * S(t) = \int_0^t F(x,\tau) S(t-\tau) d\tau \quad (65)$$

$$J_+(x,t) = G(x,t) * S(t) = \int_0^t G(x,\tau) S(t-\tau) d\tau \quad (66)$$

$$J(x,t) = H(x,t) * S(t) = \int_0^t H(x,\tau) S(t-\tau) d\tau. \quad (67)$$

Using the convolution theorem helps reduce the problem of inverting  $\Phi(x,s)$ ,  $j_+(x,s)$ , and  $j(x,s)$ , but this gain may be negated by an unwieldy convolution integral. Fortunately for certain common types of sources the gain is realized.

Although the computation of all three quantities  $\phi$ ,  $J_+$ , and  $J$  are indicated, the quantities are interrelated so that determining any two is sufficient. Usually the prime interest is in the flux, and it will be found in each case. Either the net current or the partial current will be found, depending on which is more convenient.

### B. Inversion of Transforms

To facilitate the inversion process let

$$k(s) = \left[ \frac{\tau_0}{vD} s^2 + \left( \frac{1}{vD} + \tau_0 \frac{\Sigma_a}{D} \right) s + \frac{\Sigma_a}{D} \right]^{\frac{1}{2}} = \frac{1}{c} \sqrt{(s+a)^2 - b^2}. \quad (68)$$

It is easily shown that

$$a = \frac{1}{2} \left( \frac{1}{\tau_0} + v \Sigma_a \right) = \frac{1}{2} \left( \frac{1}{\tau_0} + \frac{1}{\lambda_0} \right) \quad (69)$$

$$b = \frac{1}{2} \left( \frac{1}{\tau_0} - v \Sigma_a \right) = \frac{1}{2} \left( \frac{1}{\tau_0} - \frac{1}{\lambda_0} \right) \quad (70)$$

$$c = v \sqrt{\frac{D}{v \tau_0}} = \frac{v}{\sqrt{3}}. \quad (71)$$

It will become evident in the solution that  $c$  is the propagation velocity;  $k(s)$  is therefore called the propagation function.

### 1. Semi-infinite medium

Substituting equation 68 into 59 gives

$$f_\infty(x, s) = \frac{2(1+\tau_0 s) e^{-\frac{x}{c} \sqrt{(s+a)^2 - b^2}}}{1 + \frac{2D}{c} \sqrt{(s+a)^2 - b^2} + \tau_0 s}. \quad (72)$$

Let

$$f(x, s) = \frac{e^{-\frac{x}{c} \sqrt{(s+a)^2 - b^2}}}{1 + \frac{2D}{c} \sqrt{(s+a)^2 - b^2} + \tau_0 s} = \mathcal{L}\{F(x, t)\}. \quad (73)$$

Then

$$F_\infty(x, t) = 2F(x, t) + 2\tau_0 \frac{d}{dt} F(x, t), \quad (74)$$

providing  $F(x, 0) = 0$ , which must be so for zero initial conditions. Let

$$\rho = \sqrt{(s+a)^2 - b^2} \quad (75)$$

so that

$$f(x, p) = \frac{e^{-\frac{x}{c} p}}{1 - \tau_0 a + \frac{2D}{c} p + \tau_0 \sqrt{p^2 + b^2}} \quad (76a)$$

$$\begin{aligned} &= \frac{3b}{\tau_0} \frac{1}{p} \frac{e^{-\frac{x}{c} p}}{p + 12 \frac{bc}{v}} + \frac{2v}{\tau_0 c} \frac{e^{-\frac{x}{c} p}}{p + 12 \frac{bc}{v}} \\ &\quad - \frac{3}{\tau_0} p \frac{e^{-\frac{x}{c} p}}{(p + 12 \frac{bc}{v}) \sqrt{p^2 + b^2}} - \frac{3b^2}{\tau_0} \frac{1}{p} \frac{e^{-\frac{x}{c} p}}{(p + 12 \frac{bc}{v}) \sqrt{p^2 + b^2}}. \end{aligned} \quad (76b)$$

The complications arising from the presence of the  $\tau_0$ s terms in equation 59 is evident in equation 76. The terms in the denominator of equation 76a involving  $\tau_0$  explicitly are due to them. Without these two terms  $f(x, p)$  could have been inverted at this point, as Pasquantonio did. However, equation 76b will now be used,

The inversion of  $f(x, p)$  is made simpler by using the general formulas (18)

$$\mathcal{L}^{-1}\{e^{-as} f(s)\} = F(t-a) u(t-a) \quad (77)$$

$$\mathcal{L}^{-1}\{f(s+a)\} = e^{-at} F(t) \quad (78)$$

$$\mathcal{L}^{-1}\{sf(s)\} = F'(t) \quad (79)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s} f(s)\right\} = \int_0^t F(z) dz \quad (80)$$

where  $F(t) = \mathcal{L}^{-1}\{f(s)\}$ .

Equation 76b is now transformed inversely term by term. The first is.

$$\mathcal{L}^{-1} \left\{ \frac{e^{-\frac{x}{c} p}}{p + 12 \frac{bc}{v}} \right\} = e^{-12 \frac{bc}{v} (t - \frac{x}{c})} u(t - \frac{x}{c}). \quad (81)$$

From formula 80 it follows that

$$\mathcal{L}^{-1} \left\{ \frac{1}{p} \frac{e^{-\frac{x}{c} p}}{p + 12 \frac{bc}{v}} \right\} = \frac{v}{12 bc} \left[ 1 - e^{-12 \frac{bc}{v} (t - \frac{x}{c})} \right] u(t - \frac{x}{c}). \quad (82)$$

Employing in order formulas 77, 78, 80, and 78 converts the transform (17)

$$\mathcal{L}^{-1} \left\{ \frac{1}{\sqrt{p^2 + b^2}} \right\} = J_0(bt) \quad (83)$$

to

$$\mathcal{L}^{-1} \left\{ \frac{e^{-ap}}{(p+c)\sqrt{p^2+b^2}} \right\} = e^{-c(t-a)} \int_0^{t-a} e^{cn} J_0(b\eta) d\eta u(t-a) \quad (84)$$

and results in

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{pe^{-\frac{x}{c} p}}{(cp+12\frac{bc}{v})\sqrt{p^2+b^2}} \right\} &= J_0 \left[ b(t - \frac{x}{c}) \right] u(t - \frac{x}{c}) \\ &\quad - 12 \frac{bc}{v} e^{-12 \frac{bc}{v} (t - \frac{x}{c})} \int_0^{t-\frac{x}{c}} e^{12 \frac{bc}{v} \eta} J_0(b\eta) d\eta u(t - \frac{x}{c}). \end{aligned} \quad (85)$$

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{1}{p} \frac{e^{-\frac{x}{c} p}}{(p+12\frac{bc}{v})\sqrt{p^2+b^2}} \right\} &= \int_{\frac{x}{c}}^t e^{-12 \frac{bc}{v} (\lambda - \frac{x}{c})} \int_0^{\lambda - \frac{x}{c}} e^{12 \frac{bc}{v} \eta} J_0(b\eta) d\eta d\lambda u(t - \frac{x}{c}). \end{aligned} \quad (86)$$

The double integral of equation 86 is reduced through integration by parts to the two single integrals

$$- \frac{v}{12bc} e^{-12\frac{bc}{v}(t-\frac{x}{c})} \int_0^{t-\frac{x}{c}} e^{12\frac{bc}{v}\eta} J_0(b\eta) d\eta + \frac{v}{12bc} \int_{\frac{x}{c}}^t J_0[b(\eta - \frac{x}{c})] d\eta.$$

Therefore the inverse Laplace transform of  $f(x, p)$  with the substitution  $\tau_0 = 3D/v$  is

$$\begin{aligned} \mathcal{L}^{-1}\{f(x, p)\} &= \frac{c}{4D} u(t-\frac{x}{c}) + \frac{7c}{4D} e^{-12\frac{bc}{v}(t-\frac{x}{c})} u(t-\frac{x}{c}) - \frac{v}{D} J_0[b(t-\frac{x}{c})] u(t-\frac{x}{c}) \\ &\quad - \frac{bc}{4D} \int_{\frac{x}{c}}^t J_0[b(\eta - \frac{x}{c})] d\eta u(t-\frac{x}{c}) \\ &\quad + \frac{49bc}{4D} e^{-12\frac{bc}{v}(t-\frac{x}{c})} \int_0^{t-\frac{x}{c}} e^{12\frac{bc}{v}\eta} J_0(b\eta) d\eta u(t-\frac{x}{c}). \end{aligned} \quad (87)$$

To derive  $F(x, t)$  from  $\mathcal{L}^{-1}\{f(x, p)\}$  one must know the relationship between  $\mathcal{L}^{-1}\{f(x, p)\}$  and  $\mathcal{L}^{-1}\{f(x, s)\}$ . This formula is obtained by using formula 78 to modify the formula given by Erdélyi (18)

$$\mathcal{L}^{-1}\{f(\sqrt{s^2 - b^2})\} = F(t) + b \int_0^t F(\sqrt{t^2 - \tau^2}) I_1(b\tau) d\tau \quad (88)$$

where  $F(t) = \mathcal{L}^{-1}\{f(x, s)\}$ ; it is

$$\mathcal{L}^{-1}\{f(\sqrt{(s+a)^2 - b^2})\} = F(t) e^{-at} + b e^{-at} \int_0^t F(\sqrt{t^2 - \tau^2}) I_1(b\tau) d\tau \quad (89)$$

In this situation  $F(x, t) = \mathcal{L}^{-1}\{f(x, s)\} = \mathcal{L}^{-1}\{f(x, \sqrt{(s+a)^2 - b^2})\}$  and  $\mathcal{L}^{-1}\{f(x, p)\} = F(x, t)$ ; hence

$$\begin{aligned} F(x, t) &= \frac{c}{4D} e^{-at} u(t-\frac{x}{c}) + \frac{7c}{4D} e^{12\frac{b}{v}x - (a + 12\frac{bc}{v})t} u(t-\frac{x}{c}) \\ &\quad - \frac{v}{D} e^{-at} J_0[b(t-\frac{x}{c})] u(t-\frac{x}{c}) - \frac{bc}{4D} e^{-at} \int_{\frac{x}{c}}^t J_0[b(\lambda - \frac{x}{c})] d\lambda u(t-\frac{x}{c}) \\ &\quad + \frac{49bc}{4D} e^{12\frac{b}{v}x - (a + 12\frac{bc}{v})t} \int_0^{t-\frac{x}{c}} e^{12\frac{b}{v}\eta} J_0(b\eta) d\eta u(t-\frac{x}{c}) \end{aligned}$$

$$\begin{aligned}
& + \frac{bc}{4D} e^{-at} \int_0^t I_1(b\lambda) u(\sqrt{t^2 - \lambda^2} - \frac{x}{c}) d\lambda \\
& + \frac{7bc}{4D} e^{12\frac{b}{v}x - at} \int_0^t e^{-12\frac{bc}{v}\sqrt{t^2 - \lambda^2}} u(\sqrt{t^2 - \lambda^2} - \frac{x}{c}) I_1(b\lambda) d\lambda \\
& - \frac{bv}{D} e^{-at} \int_0^t J_0[b(\sqrt{t^2 - \lambda^2} - \frac{x}{c})] u(\sqrt{t^2 - \lambda^2} - \frac{x}{c}) I_1(b\lambda) d\lambda \\
& - \frac{b^2 c}{4D} e^{-at} \int_0^t \int_{\frac{x}{c}}^{\sqrt{t^2 - \lambda^2}} J_0[b(\eta - \frac{x}{c})] d\eta u(\sqrt{t^2 - \lambda^2} - \frac{x}{c}) I_1(b\lambda) d\lambda \\
& + \frac{49b^2 c}{4D} e^{12\frac{b}{v}x - at} \int_0^t e^{-12\frac{bc}{v}\sqrt{t^2 - \lambda^2}} \int_0^{\sqrt{t^2 - \lambda^2} - \frac{x}{c}} e^{12\frac{bc}{v}\eta} \\
& \cdot J_0(b\eta) d\eta u(\sqrt{t^2 - \lambda^2} - \frac{x}{c}) I_1(b\lambda) d\lambda.
\end{aligned} \tag{90}$$

Some elucidation of this equation is gained by making explicit the function  $u(\sqrt{t^2 - \lambda^2} - \frac{x}{c})$ . It states that  $\sqrt{t^2 - \lambda^2} - \frac{x}{c} \geq 0$ , which is equivalent to  $\lambda \leq \sqrt{t^2 - x^2/c^2}$ . Since  $\lambda$  must be real, then  $t \geq \frac{x}{c}$ .  $F(x, t)$  now takes the form

$$\begin{aligned}
F(x, t) = & \frac{c}{4D} e^{-at} u(t - \frac{x}{c}) + \frac{7c}{4D} e^{12\frac{b}{v}x - (a + 12\frac{bc}{v})t} u(t - \frac{x}{c}) \\
& - \frac{v}{D} e^{-at} J_0[b(t - \frac{x}{c})] u(t - \frac{x}{c}) - \frac{bc}{4D} e^{-at} \int_{\frac{x}{c}}^t J_0[b(\lambda - \frac{x}{c})] d\lambda u(t - \frac{x}{c}) \\
& + \frac{49bc}{4D} e^{12\frac{b}{v}x - (a + 12\frac{bc}{v})t} \int_0^{t - \frac{x}{c}} e^{12\frac{bc}{v}\lambda^2} J_0(b\lambda) d\lambda u(t - \frac{x}{c}) \\
& + \frac{bc}{4D} e^{-at} \int_0^{\sqrt{t^2 - \frac{x^2}{c^2}}} I_1(b\lambda) d\lambda u(t - \frac{x}{c}) \\
& + \frac{7bc}{4D} e^{12\frac{b}{v}x - at} \int_0^{\sqrt{t^2 - \frac{x^2}{c^2}}} e^{-12\frac{bc}{v}\sqrt{t^2 - \lambda^2}} I_1(b\lambda) d\lambda u(t - \frac{x}{c})
\end{aligned} \tag{91}$$

$$\begin{aligned}
 & -\frac{b v}{D} e^{-at} \int_0^{\sqrt{t^2 - \frac{x^2}{c^2}}} J_0 [b(\sqrt{t^2 - \lambda_3^2} - \frac{x}{c})] I_1(b\lambda_3) d\lambda_3 u(t - \frac{x}{c}) \\
 & -\frac{b^2 c}{4D} e^{-at} \int_0^{\sqrt{t^2 - \frac{x^2}{c^2}}} \int_{\frac{x}{c}}^{\sqrt{t^2 - \lambda_3^2}} J_0 [b(\eta_1 - \frac{x}{c})] d\eta_1 I_1(b\lambda_3) d\lambda_3 u(t - \frac{x}{c}) \\
 & + \frac{49 b^2 c}{4D} e^{12 \frac{b}{v} x - at} \int_0^{\sqrt{t^2 - \frac{x^2}{c^2}}} e^{-12 \frac{b c}{v} \sqrt{t^2 - \lambda_3^2}} \int_0^{\sqrt{t^2 - \lambda_3^2} - \frac{x}{c}} e^{12 \frac{b c}{v} \eta_2} \\
 & \bullet J_0(b\eta_2) d\eta_2 I_1(b\lambda_3) d\lambda_3 u(t - \frac{x}{c}).
 \end{aligned}$$

Let the variables of integration be changed according to

$$\lambda_1 = \xi/c$$

$$\lambda_2 = t - \xi/c$$

$$\lambda_3 = \sqrt{t^2 - \xi^2}/c$$

$$\eta_1 = \xi/c$$

$$\eta_2 = (\xi - x)/c.$$

Then  $F(x,t)$  becomes

$$\begin{aligned}
 F(x,t) &= \frac{c}{4D} e^{-at} u(t - \frac{x}{c}) + \frac{7c}{4D} e^{12 \frac{b}{v} x - (a + 12 \frac{b c}{v}) t} u(t - \frac{x}{c}) \\
 & - \frac{v}{D} e^{-at} J_0 [b(t - \frac{x}{c})] u(t - \frac{x}{c}) \\
 & - \frac{b}{4D} e^{-at} \int_x^{ct} [1 - 49 e^{-12 \frac{b}{v} (\xi - x)}] J_0 [\frac{b}{c} (\xi - x)] d\xi u(t - \frac{x}{c}) \\
 & + \frac{b}{4D} e^{-at} \int_x^{ct} [1 + 7 e^{-12 \frac{b}{v} (\xi - x)}] \Lambda(\xi, t) d\xi u(t - \frac{x}{c})
 \end{aligned} \tag{92}$$

$$\begin{aligned}
 & -\frac{b\nu}{Dc} e^{-at} \int_x^{ct} J_0\left[\frac{b}{c}(\xi-x)\right] \Lambda(\xi, t) d\xi u(t-\frac{x}{c}) \\
 & -\frac{b^2}{4Dc} e^{-at} \int_x^{ct} \int_x^\xi [1 - 49e^{-12\frac{b}{\nu}(\xi-\zeta)}] J_0\left[\frac{b}{c}(\xi-x)\right] d\zeta \Lambda(\xi, t) d\xi u(t-\frac{x}{c})
 \end{aligned}$$

where

$$\Lambda(\xi, t) = \frac{\xi I_1(b\sqrt{t^2 - \xi^2/c^2})}{c\sqrt{t^2 - \xi^2/c^2}}. \quad (93)$$

Finally one obtains  $F_\infty(x, t)$  by substituting equation 93 and its time derivative into equation 74. The result is (cf. Pasquantonio (9))

$$F_\infty(x, t) = \begin{cases} 0 & 0 < t < \frac{x}{c} \\ b\left(\frac{2c}{\nu}-1\right) e^{-ax/c} \delta(t-\frac{x}{c}) + \frac{3bc}{2\nu} \left[1 + \frac{bt}{2}\right] e^{-at} \\ + \frac{21bc}{2\nu} \left[1 - \frac{4\nu}{c} + \frac{bt}{2}\right] e^{-at - 12\frac{bc}{\nu}(t-\frac{x}{c})} + b e^{-at} J_1[b(t-\frac{x}{c})] \\ - 6b \left[1 + \frac{c}{4\nu} + \frac{bt}{2} - \frac{49c}{4\nu}\right] e^{-12\frac{bc}{\nu}(t-\frac{x}{c})} e^{-at} J_0[b(t-\frac{x}{c})] \\ - \frac{3b^2}{2\nu} e^{-at} \int_x^{ct} \left[1 + \frac{bt}{2} - 49e^{-12\frac{b}{\nu}(\xi-x)} - \frac{49bt}{2} e^{-12\frac{b}{\nu}(\xi-\frac{x}{c})}\right] J_0\left[\frac{b}{c}(\xi-x)\right] d\xi \quad (94) \\ + \frac{3b}{2\nu} e^{-at} \int_x^{ct} [1 + 7e^{-12\frac{b}{\nu}(\xi-x)}] [b\Lambda + \Lambda'] d\xi \\ - \frac{6b}{c} e^{-at} \int_x^{ct} J_0\left[\frac{b}{c}(\xi-x)\right] [b\Lambda + \Lambda'] d\xi \\ - \frac{3b^2}{2\nu c} e^{-at} \int_x^{ct} \int_x^\xi [1 - 49e^{-12\frac{b}{\nu}(\xi-\zeta)}] J_0\left[\frac{b}{c}(\xi-x)\right] d\zeta [b\Lambda + \Lambda'] d\xi \\ t \geq \frac{x}{c} \end{cases}$$

Rather than transform  $h_\infty(x, s)$  to find the current directly, it is more convenient to transform  $g_\infty(x, s)$  due to its simplicity and obtain  $J(x, t)$  from

$$J(x, t) = 2 J_+(x, t) - \frac{1}{2} \phi(x, t). \quad (22)$$

Starting with

$$g_\infty(x, s) = \frac{1}{2} e^{-k(s)x} = \mathcal{L}\{G_\infty(x, t)\} \quad (60)$$

and substituting equation 68 for  $k(s)$  gives

$$g_\infty(x, s) = \frac{1}{2} e^{-\frac{x}{c} \sqrt{(s+a)^2 - b^2}}. \quad (95)$$

Combining equations 75 and 95 results in

$$g(x, p) = \frac{1}{2} e^{-\frac{x}{c} p} = \mathcal{L}\{G(x, t)\}. \quad (96)$$

The inverse transform of this is

$$G(x, t) = \frac{1}{2} \delta(t - \frac{x}{c}). \quad (97)$$

Formula 89 provides the step from  $G(x, t)$  to  $G_\infty(x, t)$ . The result is

$$G_\infty(x, t) = \frac{1}{2} e^{-\frac{a}{c} x} \delta(t - \frac{x}{c}) + \frac{b}{2} e^{-at} I_1(b \sqrt{t^2 - \frac{x^2}{c^2}}) u(t - \frac{x}{c}). \quad (98)$$

Convoluting 94 and 98 with the source  $S(t)$  according to 65 and 66 produces  $\phi(x, t)$  and  $J_+(x, t)$ . For the unit impulse source and the unit step source the fluxes and currents are

i.  $S(t) = \delta(t)$

$$\phi_{\infty}(x, t) = \int_0^t F_{\infty}(x, \tau) \delta(t-\tau) d\tau = F_{\infty}(x, t) \quad (99)$$

$$J_{+_{\infty}}(x, t) = \int_0^t G_{\infty}(x, \tau) \delta(t-\tau) d\tau = G_{\infty}(x, t) \quad (100)$$

$$J_{\infty}(x, t) = \int_0^t H_{\infty}(x, \tau) \delta(t-\tau) d\tau = H_{\infty}(x, t) \quad (101)$$

ii.  $S(t) = u(t)$

$$\phi_{\infty}(x, t) = \int_0^t F_{\infty}(x, \tau) u(t-\tau) d\tau = \int_0^t F_{\infty}(x, \tau) d\tau \quad (102)$$

$$J_{+_{\infty}}(x, t) = \int_0^t G_{\infty}(x, \tau) u(t-\tau) d\tau = \int_0^t G_{\infty}(x, \tau) d\tau \quad (103)$$

$$J_{\infty}(x, t) = \int_0^t H_{\infty}(x, \tau) u(t-\tau) d\tau = \int_0^t H_{\infty}(x, \tau) d\tau. \quad (104)$$

## 2. Infinite slab

The flux  $\phi_s(x, t) = \mathcal{L}^{-1}\{\Phi_s(x, t)\}$  will be found first. The function  $f_s(x, s)$  is similar to  $f_{\infty}(x, s)$ , differing chiefly by the factor

$$\left[ 1 - \frac{1 - 2Dk(s) + \tau_0 s}{1 + 2Dk(s) + \tau_0 s} e^{-2hk(s)} \right]^{-1}.$$

This function can be expanded according to the series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \quad x^2 < 1 \quad (105)$$

resulting in

$$\left[ 1 - \frac{1 - 2Dk(s) + \tau_0 s}{1 + 2Dk(s) + \tau_0 s} e^{-2hk(s)} \right]^{-1} = \sum_{n=0}^{\infty} \left[ \frac{1 - 2Dk(s) + \tau_0 s}{1 + 2Dk(s) + \tau_0 s} \right]^n e^{-2nhk(s)} \quad (106a)$$

$$= \sum_{n=0}^{\infty} r^n(s) e^{-2nhk(s)} \quad (106b)$$

where

$$r(s) = \frac{1 - 2Dk(s) + \tau \cdot s}{1 + 2Dk(s) + \tau \cdot s} \quad (107)$$

if the condition

$$|r(s) e^{-2hk(s)}| < 1 \quad (108)$$

holds. This condition will now be examined. Explicitly,

$$|r(s) e^{-2hk(s)}| = \left| \frac{1 - \frac{2D}{c} \sqrt{(s+a)^2 - b^2} + \tau \cdot s}{1 + \frac{2D}{c} \sqrt{(s+a)^2 - b^2} + \tau \cdot s} e^{-\frac{2h}{c} \sqrt{(s+a)^2 - b^2}} \right|. \quad (109)$$

Since  $a > b$  (see equations 69 and 70), then  $\sqrt{(s+a)^2 - b^2} > 0$  for all non-negative real values of  $s$ . Furthermore for all  $s$  with a non-negative real part the complex number given by  $z = (s+a)^2 - b^2$  will have the argument  $-\pi < \theta < \pi$ ; hence,  $\sqrt{z} = \sqrt{(s+a)^2 - b^2}$  must have an argument  $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ , for which  $\cos \theta > 0$ . Therefore  $\sqrt{(s+a)^2 - b^2}$  has a positive real part and it follows that condition 108 holds in the region of interest for the inversion integral.

Manipulating equation 56 after simplifying with equations 106b and 48 results in

$$\Phi_s(x, s) = \sum_{n=0}^{\infty} r^n(s) [\Phi_{\infty}(2nh+x, s) - \Phi_{\infty}(2(n+1)h-x, s)]. \quad (110)$$

In the same manner from equations 48 and 40 one finds

$$j_s(x, s) = \sum_{n=0}^{\infty} r^n(s) [j_{\infty}(2nh+x, s) + j_{\infty}(2(n+1)h-x, s)]. \quad (111)$$

Since  $\phi_\infty(x,t)$  and  $J_\infty(x,t)$  are already known, it suffices to determine  $R(t)$  to find  $\phi_s(x,t)$  and  $J_s(x,t)$ .

Rearranging equation 107 produces

$$r(s) = 1 + \frac{4c}{D} \frac{(1 + \frac{3D}{\nu}s)\sqrt{(s+\alpha)^2 - b^2}}{s^2 + \frac{2\nu}{3D}(1 - \frac{12Db}{\nu})s + (\frac{\nu}{3D})^2(1 - \frac{24Db}{\nu})} \quad (112a)$$

$$= 1 + \frac{4c}{D} \frac{(1 + \frac{3D}{\nu}s)\sqrt{(s+\alpha)^2 - b^2}}{(s+\alpha)^2 - \beta^2} \quad (112b)$$

where

$$\alpha = \frac{1}{\tau_0} - 4b \quad (113)$$

$$\beta = 4b. \quad (114)$$

(It is interesting to notice that constants in the simplified quadratic expressions in equation 112 have a certain similarity of form:

$$\begin{aligned} \alpha + b &= \frac{1}{\tau_0} \\ \alpha + \beta &= \frac{1}{\tau_0} \end{aligned}$$

Perhaps more intriguing is that this similarity is based on  $\tau_0$ .) A more useful form of 112b for the use of tables of inverse Laplace transforms is

$$\begin{aligned} r(s) &= 1 + \frac{2c}{\beta D} \left[ \frac{3D}{\nu} s^3 + (1 + \frac{6D\alpha}{\nu}) s^2 + (\frac{3D\alpha^2}{\nu} + 2\alpha - \frac{3Db^2}{\nu}) s \right. \\ &\quad \left. + \alpha^2 - b^2 \right] \left[ \frac{1}{(s+\alpha-\beta)\sqrt{(s+\alpha)^2 - b^2}} - \frac{1}{(s+\alpha+\beta)\sqrt{(s+\alpha)^2 - b^2}} \right]. \end{aligned} \quad (115)$$

Applying the general formulas 78, 80, and 78 in that order to each of the fractions in the second line of equation 115, one can take advantage of

the formula (18)

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s^2 - a^2}}\right\} = I_0(at). \quad (116)$$

The result is

$$\begin{aligned} & \mathcal{L}^{-1}\left\{\frac{1}{(p+a \pm \beta)\sqrt{(p+a)^2 - b^2}}\right\} \\ &= e^{-(\alpha \pm \beta)t} \int_0^t e^{-(a - \alpha \mp \beta)\lambda} I_0(b\lambda) d\lambda. \end{aligned} \quad (117)$$

In consequence of equation 117

$$\begin{aligned} R(t) &= S(t) + \frac{84bc}{v} e^{-at} I_0(bt) - \frac{12bc}{v} e^{-at} I_1(bt) \\ &+ \frac{384b^2c}{v} e^{-at} \int_0^t e^{-7b(\lambda-t)} I_0(b\lambda) d\lambda. \end{aligned} \quad (118)$$

The knowledge of  $R(t)$  provides the key to the solution for the infinite slab. In the time domain the product  $r^n(s) \Phi_\infty(x, s)$  becomes the n-fold convolution of  $R(t)$  with  $\phi_\infty(x, t)$ . The infinite series is truncated due to the restriction  $\phi_\infty(x, t) = 0$  for  $t < \frac{x}{c}$ . It follows that  $\phi_\infty(2nh \pm x, t)$  is zero for  $t < \frac{2nh \pm x}{c}$  or  $n > \frac{ct \mp x}{2h}$ . Thus the solution for the infinite slab is given by

$$\phi_s(x, t) = \sum_{n=0}^{N_1} R^n(t) * \phi_\infty(2nh \pm x, t) - \sum_{n=0}^{N_2} R^n(t) * \phi_\infty[2(n+1)h \pm x, t] \quad (119)$$

$$J_s(x, t) = \sum_{n=0}^{N_1} R^n(t) * J_\infty(2nh \pm x, t) + \sum_{n=0}^{N_2} R^n(t) * J_\infty[2(n+1)h \pm x, t] \quad (120)$$

where  $N_1$  and  $N_2$  are the largest integers such that

$$N_1 \leq \frac{ct - x}{2h} \quad (121)$$

$$N_2 \leq \frac{ct - (2h-x)}{2h}. \quad (122)$$

The need for two series of terms with pairing of powers of  $R^n(t)$  is to satisfy the boundary condition of zero flux at  $x = h$ . The position  $x$  is mirrored at each multiple of  $h$  (see Fig. 1) so that at  $x = h$  the two series will cancel. In Fig. 2 the path of the wave front is diagrammed.

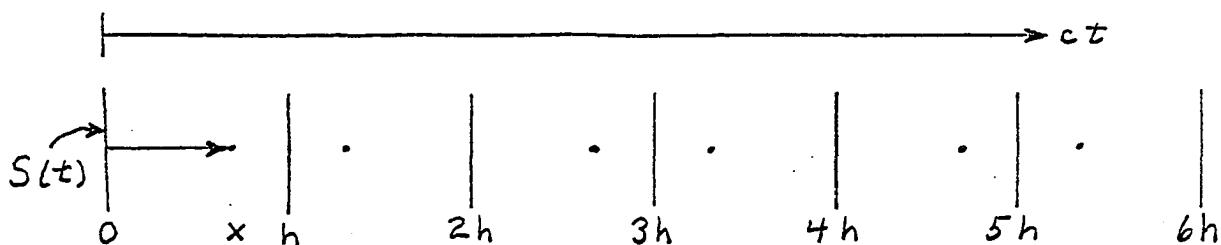


Fig. 1. Image representation of the point  $(x, t)$

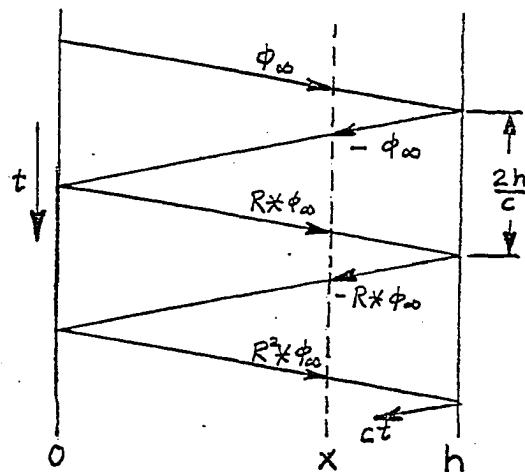


Fig. 2. Traveling wave representation of the point  $(x, t)$

The front travels a distance  $ct$  in time  $t$ . Each time the path crosses  $x$  another term is required to describe the flux at that point. Fortunately the attenuation of the flux is high and only a few terms are required. This form of the solution is therefore most useful in the study

of transients; it becomes quite clumsy and complicated in the asymptotic region. Since the telegrapher's and diffusion equations give the same results in the asymptotic sense, the method of solution followed here would not be required, and a simpler expression is obtained for the flux.

The magnitude of the total flux is reduced in two ways. The first cause for reduction is the attenuation of the flux due to scattering and absorption. The attenuation coefficient  $\alpha = (\nu/3D + \nu\Sigma_a)/2$  is very large. For an absorptive moderator such as water  $\alpha = 2.34 \times 10^5 \text{ sec}^{-1}$ ; for a low absorber like heavy water  $\alpha = 0.38 \times 10^5 \text{ sec}^{-1}$ . Hence the flux is attenuated by a factor of  $e$  within 4.3 microseconds in water and within 17 microseconds in heavy water. The second effect is the loss sustained upon reflection at the boundaries. At the origin it is by the convolution with  $R(t)$ . At  $x=h$  it is by the magnitude of the front incident to the boundary.

From these results one concludes that the neutron flux does exhibit a wavelike character, exemplified by the case of the infinite slab. A well-defined wavefront exists in the condition that  $\phi_\infty(x,t)=0$  for  $t < \frac{x}{c}$  and a residual effect persists at each point crossed by this wave front. The multitude of convolution integrals discloses the diffusion character of the flux. Thus the neutron flux, diffusive in the asymptotic state, becomes wavelike in the transient state.

The extension of this solution to include multiplying media is accomplished by replacing  $S(x,t)$  in equation 15 by  $S(x,t) + \nu\Sigma_f\phi(x,t)$ . The equivalent result is obtained by leaving  $S(x,t)$  alone and replacing  $\Sigma_a$  by  $\Sigma_a - \nu\Sigma_f$ . The major effect is that the attenuation is decreased

by  $e^{Vv \Sigma_f t/2}$ .

### C. Special Case: Sinusoidal Source

A sinusoidal source  $S(t) = S \cos \omega t$ . (actually one must consider the source  $S(t) = S \cos \omega t + S_0$  where  $S_0 \geq S$  to meet the physical requirement of a non-negative flux) is considered. A well-known technique (19) for finding the steady-state solution of the flux or of some other physical variable for such a source is to Laplace transform the flux, substitute  $i\omega$  ( $\omega$  = frequency) for the transform variable  $s$ , and replace the transformed source  $\sigma(s)$  by  $Se^{i\omega t}$ . In doing so the source under consideration is  $S(t) = Se^{i\omega t}$ ; only the real part of the result has meaning. The sinusoidal source may be considered separately from the constant part  $S_0$  since the two are linear and their results additive.

Equation 36 becomes

$$k(i\omega) = \left[ \frac{\Sigma_a}{D} - \frac{3}{v^2} \omega^2 + i \frac{1+3D\Sigma_a}{vD} \omega \right]^{\frac{1}{2}} \quad (123a)$$

$$= \alpha + i\beta = \gamma. \quad (123b)$$

Squaring 123 and equating real and imaginary parts gives us

$$\alpha = \left[ \frac{\Sigma_a}{2D} - \frac{3}{2} \left( \frac{\omega}{v} \right)^2 + \frac{1}{2} \sqrt{9 \left( \frac{\omega}{v} \right)^4 + (1+9D^2\Sigma_a^2) \left( \frac{\omega}{vD} \right)^2 + \left( \frac{\Sigma_a}{D} \right)^2} \right]^{\frac{1}{2}} \quad (124)$$

$$\beta = \left[ -\frac{\Sigma_a}{2D} + \frac{3}{2} \left( \frac{\omega}{v} \right)^2 + \frac{1}{2} \sqrt{9 \left( \frac{\omega}{v} \right)^4 + (1+9D^2\Sigma_a^2) \left( \frac{\omega}{vD} \right)^2 + \left( \frac{\Sigma_a}{D} \right)^2} \right]^{\frac{1}{2}}. \quad (125)$$

Due to the nature of solution 37 the sign of the square root in equation 123a and the outer square root in equation 124 must be chosen so that  $\alpha$  is positive. The inner square root in equations 124 and 125 must be taken

in the positive sense since  $\alpha$  and  $\beta$  are real.

### 1. Semi-infinite medium

From equations 48, 49, and 50 for the semi-infinite medium

$$\phi_{\infty}^{\circ}(x,t) = S \frac{2(1+i\tau_0\omega)}{1+2Dk(i\omega)+i\tau_0\omega} e^{-k(i\omega)x+i\omega t} \quad (126)$$

$$J_{+\infty}^{\circ}(x,t) = \frac{S}{2} e^{-k(i\omega)x+i\omega t} \quad (127)$$

$$J_{-\infty}^{\circ}(x,t) = S \frac{2Dk(i\omega)}{1+2Dk(i\omega)+i\tau_0\omega} e^{-k(i\omega)x+i\omega t} \quad (128)$$

where the superscript  $\circ$  refers to the steady-state value. The polar form of  $\gamma$  is

$$\gamma = |\gamma| e^{i\varepsilon} \quad (129a)$$

where

$$|\gamma| = \sqrt{\alpha^2 + \beta^2}; \quad \cos \varepsilon = \frac{\alpha}{|\gamma|}; \quad \sin \varepsilon = \frac{\beta}{|\gamma|} \quad (129b)$$

and  $\gamma_1$  and  $\gamma_2$  are defined by

$$\gamma_1 = 1 + i\tau_0\omega \quad (130a)$$

$$\gamma_2 = 1 + 2D\alpha + i(\tau_0\omega + 2D\beta) \quad (131a)$$

where

$$|\gamma_1| = \sqrt{1 + \tau_0^2 \omega^2}; \quad \cos \varepsilon_1 = \frac{1}{|\gamma_1|}; \quad \sin \varepsilon_1 = \frac{\tau_0 \omega}{|\gamma_1|} \quad (130b)$$

$$|\gamma_2| = \sqrt{(1+2D\alpha)^2 + (\tau_0\omega + 2DB)^2}$$

(131b)

$$\cos \varepsilon_2 = \frac{1+2D\alpha}{|\gamma_2|}; \quad \sin \varepsilon_2 = \frac{\tau_0\omega + 2DB}{|\gamma_2|}.$$

Then equations 126, 127, and 128 take the forms

$$\phi_{\infty}^o(x,t) = 2S \frac{|\gamma_1|}{|\gamma_2|} e^{-\alpha x + i(\omega t - \beta x + \varepsilon_1 - \varepsilon_2)} \quad (132)$$

$$J_{+o}^o(x,t) = \frac{S}{2} e^{-\alpha x + i(\omega t - \beta x)} \quad (133)$$

$$J_{oo}^o(x,t) = 2DS \frac{|\gamma_1|}{|\gamma_2|} e^{-\alpha x + i(\omega t - \beta x + \varepsilon - \varepsilon_2)} \quad (134)$$

Hence the steady state values are

$$\phi_{\infty}^o(x,t) = 2S \frac{|\gamma_1|}{|\gamma_2|} e^{-\alpha x} \cos(\omega t - \beta x + \varepsilon_1 - \varepsilon_2) \quad (135)$$

$$J_{+o}^o(x,s) = \frac{S}{2} e^{-\alpha x} \cos(\omega t - \beta x) \quad (136)$$

$$J_{oo}^o(x,s) = 2DS \frac{|\gamma_1|}{|\gamma_2|} e^{-\alpha x} \cos(\omega t - \beta x + \varepsilon - \varepsilon_2). \quad (137)$$

All of these functions suffer the same attenuation  $e^{-\alpha x}$  but their phase shifts differ. Note that only the partial current is in phase with the source at the origin; the flux and net current are initially delayed by  $\frac{\varepsilon_2 - \varepsilon_1}{\omega}$  and  $\frac{\varepsilon_2 - \varepsilon}{\omega}$  respectively. This is due to the direct relationship between the partial current and source at the origin and the more complicated relationships for the flux and net current. The following definitions are made

$\alpha$  = attenuation constant

$\beta$  = phase constant

$\gamma$  = propagation constant or complex inverse relaxation length

At the points where the delays  $(\beta x - \varepsilon_1 + \varepsilon_2)/\omega$ ,  $\beta x/\omega$  and  $(\beta x - \varepsilon_1 - \varepsilon_2)/\omega$  are multiples of the period  $T = 2\pi/\omega$ , the functions  $\phi_s^o(x,t)$ ,  $J_{+s}^o(x,t)$ , and  $J_s^o(x,t)$  are in phase with themselves. These points occur at distances of the wave length  $\lambda = 2\pi/\beta$ . The wave velocity is then given by  $\lambda/T = \omega/\beta$ . Although  $\alpha$  and  $\beta$  are constants with respect to space and time, they vary with frequency. This dependence is displayed in the next section in connection with a comparison with diffusion theory.

## 2. Infinite slab

Following the same procedure used in the preceding case equations 56, 57, and 58 are put into the form

$$\phi_s^o(x,t) = 2S \frac{|\gamma_1|}{|\gamma_3|} \left[ e^{\alpha(h-x) + i\beta(h-x)} - e^{-\alpha(h-x) - i\beta(h-x)} \right] e^{i(\omega t + \varepsilon_1 - \varepsilon_3)} \quad (138)$$

$$J_{+s}^o(x,t) = \frac{S}{2|\gamma_3|} \left[ |\gamma_2| e^{\alpha(h-x) + i(\beta h - \beta x + \varepsilon_2)} - |\gamma_4| e^{-\alpha(h-x) - i(\beta h - \beta x - \varepsilon_4)} \right] e^{i(\omega t - \varepsilon_3)} \quad (139)$$

$$\begin{aligned} j_s^o(x,t) = & \frac{2DS|\gamma|}{|\gamma_3|} \left[ e^{\alpha(h-x) + i\beta(h-x)} \right. \\ & \left. + e^{-\alpha(h-x) - i\beta(h-x)} \right] e^{i(\omega t + \varepsilon - \varepsilon_3)} \end{aligned} \quad (140)$$

where

$$\gamma_3 = \gamma_2 e^{hk(i\omega)} - \gamma_4 e^{-hk(i\omega)} \quad (141)$$

$$\gamma_4 = 1 - 2D\alpha + i(\tau_0\omega - 2DB). \quad (142)$$

The addition of another boundary prevents one from writing such simple expressions for the damping and phase shifting as for the semi-infinite medium. Taking the real parts of equations 138 and 141, gives

$$\phi_s^o(x,t) = 4S \frac{|\gamma_1|}{|\gamma_3|} \sinh \alpha(h-x) \cos \beta(h-x) \cos (\omega t + \varepsilon_1 - \varepsilon_3) \quad (143)$$

$$j_s^o(x,t) = 4DS \frac{|\gamma|}{|\gamma_3|} \cosh \alpha(h-x) \cos \beta(h-x) \cos (\omega t + \varepsilon - \varepsilon_3). \quad (144)$$

Now two independent criteria must be satisfied for phase coincidence:

both  $\beta(h-x)$  and  $\varepsilon_3 - \varepsilon_1$  or  $\varepsilon_3 - \varepsilon$ , as the case may be, must be multiples of  $2\pi$ .

#### D. Comparison with Diffusion Theory

The purpose of this section is to present comparisons between the author's results and similar results obtained from diffusion theory. The first comparison, between the basic equations, was a part of chapter III. In this section the Laplace transformed equations are compared first. Then the solution for the flux for zero initial conditions will be derived

from the diffusion equation. A qualitative study of the solutions utilizing an analog computer is presented. The steady-state values for a sinusoidal source will also be compared, along with a frequency analysis of the attenuation and phase constants.

In this section the same notation as in the foregoing equations will be used with the understanding that it now refers to the diffusion equation. Where direct comparisons with telegrapher's and diffusion equations dictate a need for clarification the subscripts  $t$  and  $d$  will be used.

The diffusion equation, that is, equation 17, is Laplace transformed to yield

$$\frac{s}{v} \bar{\Phi}(x,s) = D \bar{\Phi}''(x,s) - \sum_a \bar{\Phi}(x,s) + \sigma(x,s) \quad (145)$$

for zero initial conditions. In the absence of internal sources

$$\bar{\Phi}''(x,s) - k^2(s) \bar{\Phi}(x,s) = 0. \quad (146)$$

where

$$k^2(s) = \frac{1}{vD} s + \frac{\sum_a}{D}. \quad (147)$$

Again the general solution is

$$\bar{\Phi}(x,s) = A(s) e^{-k(s)x} + B(s) e^{k(s)x}. \quad (37)$$

However in this case

$$J(x,t) = -D \frac{\partial}{\partial x} \phi(x,t) \quad (148)$$

$$J_d(x,t) = \frac{1}{4} \phi(x,t) - \frac{D}{2} \frac{\partial}{\partial x} \phi(x,t) \quad (149)$$

so that

$$j(x,s) = D k(s) [A(s) e^{-k(s)x} - B(s) e^{k(s)x}] \quad (150)$$

$$j_+(x,s) = \frac{1+2Dk(s)}{4} A(s) e^{-k(s)x} + \frac{1-2Dk(s)}{4} B(s) e^{k(s)x}. \quad (151)$$

The boundary conditions are the same as before except that the different definition of current changes condition 44 to

$$\frac{1}{2} \bar{\Phi}(0,s) - D \bar{\Phi}'(0,s) = \sigma(s). \quad (152)$$

The mathematical procedure in finding  $\bar{\Phi}$ ,  $j$ , and  $j_+$  for the two cases is identical to that in section A. For these results and the comparison, refer to Table 1. Note that the consistent  $P_1$  approximation (telegrapher's equation) degenerates to the diffusion approximation for  $\tau_0 = 0$ .

In Table 1

$$\bar{\Phi}_\infty(x,s) = \frac{2e^{-k(s)x}}{1+2Dk(s)} \sigma(s) = f_\infty(x,s) \sigma(s). \quad (153)$$

The function  $f_\infty(x,s)$  can be written as

$$f_\infty(x,s) = \sqrt{\frac{V}{D}} \frac{e^{-\frac{x}{\sqrt{V}D} \sqrt{s+v\Sigma_a}}}{\frac{1}{2} \sqrt{\frac{V}{D}} + \sqrt{s+v\Sigma_a}}. \quad (154)$$

Erdélyi (18) lists the formula

$$\mathcal{L}^{-1} \left\{ \frac{e^{-\alpha \sqrt{p}}}{B + \sqrt{p}} \right\} = \frac{1}{\sqrt{\pi t}} e^{-\alpha^2/4t} - \beta e^{\alpha B + \beta^2 t} \operatorname{Erfc} \left( \frac{\alpha}{2\sqrt{t}} + \beta \sqrt{t} \right) \quad (155)$$

Table 1. Comparison of the Laplace transforms of the solutions

	Consistent P <sub>1</sub> approximation	Diffusion approximation
$\Phi_\infty(x, s)$	$\frac{2(1 + \tau_0 s)}{1 + 2Dk(s) + \tau_0 s} \sigma(s) e^{-k(s)x}$	$\frac{2}{1 + 2Dk(s)} \sigma(s) e^{-k(s)x}$
$j_{+\infty}(x, s)$	$\frac{1}{2} \sigma(s) e^{-k(s)x}$	$\frac{1}{2} \sigma(s) e^{-k(s)x}$
$j_\infty(x, s)$	$\frac{2Dk(s)}{1 + 2Dk(s) + \tau_0 s} \sigma(s) e^{-k(s)x}$	$\frac{2Dk(s)}{1 + 2Dk(s)} \sigma(s) e^{-k(s)x}$
$\Phi_s(x, s)$	$\frac{2(1 + \tau_0 s)}{1 + 2Dk(s) + \tau_0 s} \frac{e^{-k(s)x} - e^{-(2h-x)k(s)}}{1 - \frac{1 - 2Dk(s) + \tau_0 s}{1 + 2Dk(s) + \tau_0 s} e^{-2hk(s)}} \sigma(s)$	$\frac{2}{1 + 2Dk(s)} \frac{e^{-k(s)x} - e^{-(2h-x)k(s)}}{1 - \frac{1 - 2Dk(s)}{1 + 2Dk(s)} e^{-2hk(s)}} \sigma(s)$
$j_{+s}(x, s)$	$\frac{1}{2} \frac{e^{-k(s)x} - \frac{1 - 2Dk(s) + \tau_0 s}{1 + 2Dk(s) + \tau_0 s} e^{-(2h-x)k(s)}}{1 - \frac{1 - 2Dk(s) + \tau_0 s}{1 + 2Dk(s) + \tau_0 s} e^{-2hk(s)}} \sigma(s)$	$\frac{1}{2} \frac{e^{-k(s)x} - \frac{1 - 2Dk(s)}{1 + 2Dk(s)} e^{-(2h-x)k(s)}}{1 - \frac{1 - 2Dk(s)}{1 + 2Dk(s)} e^{-2hk(s)}} \sigma(s)$
$j_s(x, s)$	$\frac{2Dk(s)}{1 + 2Dk(s) + \tau_0 s} \frac{e^{-k(s)x} + e^{-(2h-x)k(s)}}{1 - \frac{1 - 2Dk(s) + \tau_0 s}{1 + 2Dk(s) + \tau_0 s} e^{-2hk(s)}} \sigma(s)$	$\frac{2Dk(s)}{1 + 2Dk(s)} \frac{e^{-k(s)x} + e^{-(2h-x)k(s)}}{1 - \frac{1 - 2Dk(s)}{1 + 2Dk(s)} e^{-2hk(s)}} \sigma(s)$
$k(s)$	$\left[ \frac{\tau_0}{vD} s^2 + \frac{1 + \tau_0 v \sum_a}{vD} s + \frac{\sum_a}{D} \right]^{\frac{1}{2}}$	$\left[ \frac{1}{vD} s + \frac{\sum_a}{D} \right]^{\frac{1}{2}}$

which by utilizing formula 78 is altered to

$$\mathcal{L}^{-1} \left\{ \frac{e^{-\alpha \sqrt{p+\gamma}}}{\beta + \sqrt{p+\gamma}} \right\} = \frac{1}{\sqrt{\pi t}} e^{-(\frac{\alpha^2}{4t} + \gamma t)} - \beta e^{\alpha \beta + (\beta^2 - \gamma)t} \operatorname{Erfc} \left( \frac{\alpha}{2\sqrt{t}} + \beta \sqrt{t} \right). \quad (156)$$

The function  $\operatorname{Erfc}(y)$  is the complimentary error function and is defined by

$$\operatorname{Erfc}(y) = \frac{2}{\sqrt{\pi}} \int_y^\infty e^{-t^2} dt = 1 - \operatorname{Erf}(y) = 1 - \frac{2}{\sqrt{\pi}} \int_0^y e^{-t^2} dt. \quad (157)$$

Thus the inverse Laplace transform of  $f_\infty(x, s)$  is (cf. equation 94)

$$F_\infty(x, t) = \sqrt{\frac{v}{\pi D t}} e^{-\left(\frac{x^2}{4vDt} + v\Sigma_a t\right)} - \frac{v}{2D} e^{\frac{x}{2D} + \left(\frac{v}{4D} - v\Sigma_a\right)t} \operatorname{Erfc} \left[ \frac{x}{2\sqrt{vDt}} + \frac{1}{2} \sqrt{\frac{vt}{D}} \right]. \quad (158)$$

In Table 1

$$j_{+\infty}(x, s) = \frac{1}{2} \sigma(s) e^{-k(s)x} = g_\infty(x, s) \sigma(s) \quad (159)$$

where

$$g_\infty(x, s) = \frac{1}{2} e^{-\frac{x}{\sqrt{vD}} \sqrt{s+v\Sigma_a}}. \quad (160)$$

The convenient formula for inverting  $g_\infty(x, s)$  is (18)

$$\mathcal{L}^{-1} \left\{ e^{-\sqrt{\alpha p}} \right\} = \frac{1}{2} \sqrt{\frac{\alpha}{\pi}} t^{-\frac{3}{2}} e^{-\frac{\alpha}{4t}} \quad (161)$$

which is easily altered to

$$\mathcal{L}^{-1} \left\{ e^{-\sqrt{\alpha(p+\beta)}} \right\} = \frac{1}{2} \sqrt{\frac{\alpha}{\pi}} t^{-\frac{3}{2}} e^{-\frac{\alpha}{4t} - \beta t} \quad (162)$$

by utilizing formula 78. Therefore (cf. equation 98)

$$G_\infty(x,t) = \frac{x^2}{4\sqrt{\pi} v D} t^{-\frac{3}{2}} e^{-\frac{x^2}{4vD} - v \sum_a t} \quad (163)$$

Again the convolution of  $F_\infty$  and  $G_\infty$  with the source is used to calculate  $\phi_\infty$  and  $J_\infty$ . Also, for the special cases  $S(t) = \delta(t)$  and  $S(t) = u(t)$ , equations 99 - 104 can be used if  $F_\infty$  and  $G_\infty$  are taken from this section.

A qualitative comparison of the fluxes for  $S(t) = u(t)$  is shown in Fig. 3. The limitations of the Pace TR-10 analog computer prevents one from making a comparison for  $S(t) = \delta(t)$ ; the correct boundary condition could not be placed on the analog for the telegrapher's equation. The computer only approximates a step change; consequently the well-defined wave front is rounded off. The proper minimum cutoff points are shown by the vertical lines. The effect of the source is felt immediately throughout the diffusion medium. The additional absorption of the front before it arrives at a given point lowers the asymptotic flux in the telegrapher's medium from that of the diffusion medium. The effect of the front is to increase the slope of the initial rise of the flux as well as to delay the rise. The first of these last two effects is the more prominent in Fig. 3.

One observes in Table 1 that  $\bar{\phi}_s(x,s)$  is related to  $\bar{\phi}_\infty(x,s)$  in exactly the same way in the diffusion approximation as in the consistent  $P_1$  approximation. Likewise  $j_s(x,s)$  and  $j_\infty(x,s)$  are so related. Hence equations 119 and 120 also hold for the diffusion approximation, where  $r(s)$  is now defined by

$$r(s) = \frac{2 - D k(s)}{2 + D k(s)} \quad (164a)$$

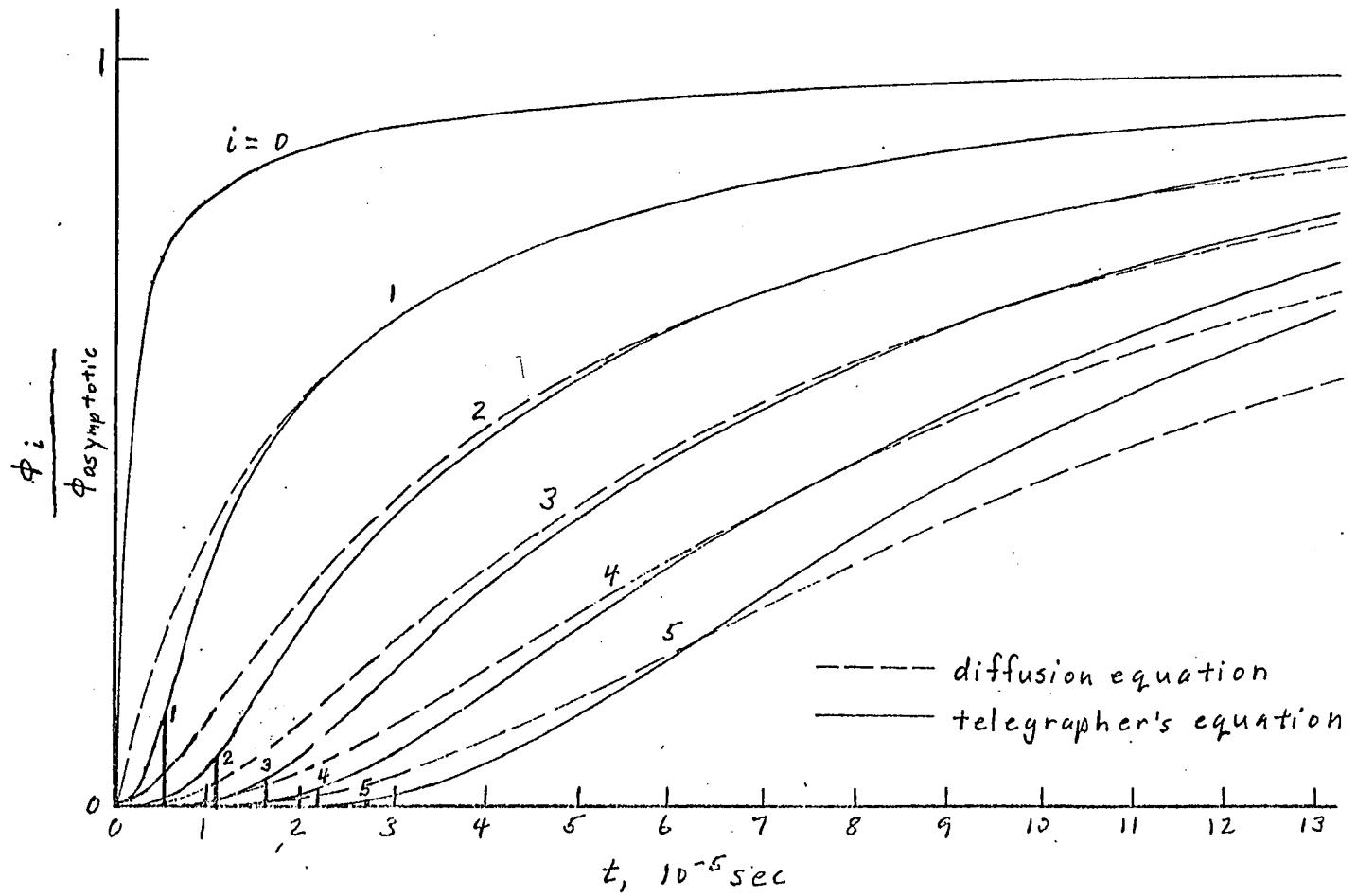


Fig. 3. Comparison of fluxes for semi-infinite medium where  $S(t) = u(t)$ . The flux  $\phi_i$  corresponds to the position  $x = iL/4$  where  $L$  is the diffusion length. In this case the medium is water.

$$= -1 + \sqrt{\frac{v}{D}} \frac{1}{\frac{1}{2}\sqrt{\frac{v}{D}} + \sqrt{s + v\Sigma_a}}. \quad (164b)$$

Once again reference is made to Erdélyi (18) for a formula; this time it is

$$\mathcal{L}^{-1} \left\{ \frac{1}{\alpha + \sqrt{p}} \right\} = \frac{1}{\sqrt{\pi t}} - \alpha e^{\alpha^2 t} \operatorname{Erfc}(\alpha\sqrt{t}) \quad (165)$$

which is converted to

$$\mathcal{L}^{-1} \left\{ \frac{1}{\alpha + \sqrt{p + \beta}} \right\} = \frac{1}{\sqrt{\pi t}} e^{-\beta t} - \alpha e^{(\alpha^2 - \beta)t} \operatorname{Erfc}(\alpha\sqrt{t}) \quad (166)$$

by means of formula 78. The resulting inversion of  $r(s)$  is (cf. 118)

$$R(t) = -\delta(t) + \sqrt{\frac{v}{\pi D t}} e^{-v\Sigma_a t} - \frac{v}{2D} e^{(\frac{v}{4D} - v\Sigma_a)t} \operatorname{Erfc}(\frac{1}{2}\sqrt{\frac{v t}{D}}). \quad (167)$$

Now the special case of the sinusoidal source is considered. From equation 147

$$k(i\omega) = \left( \frac{\Sigma_a}{D} + i \frac{\omega}{vD} \right)^{\frac{1}{2}} = \gamma = \alpha + i\beta. \quad (168)$$

It can be shown that

$$\alpha = \left[ \frac{\Sigma_a}{2D} + \frac{1}{2} \sqrt{\left( \frac{\Sigma_a}{D} \right)^2 + \left( \frac{\omega}{vD} \right)^2} \right]^{\frac{1}{2}} \quad (169)$$

$$\beta = \left[ -\frac{\Sigma_a}{D} + \frac{1}{2} \sqrt{\left( \frac{\Sigma_a}{D} \right)^2 + \left( \frac{\omega}{vD} \right)^2} \right]^{\frac{1}{2}}. \quad (170)$$

These parameters will now be compared with  $\alpha_t$  and  $\beta_t$ , given by equations 124 and 125. For  $\omega = 0$

$$\alpha_d = \alpha_t = \sqrt{\frac{\Sigma_a}{D}} = \frac{1}{L} \quad \beta_d = \beta_t = 0.$$

It follows that the ratio of the attenuation constants for  $\omega = 0$  is unity; however, although both phase constants are zero, their ratio, instead of being unity, is given by the limit

$$\lim_{\omega \rightarrow 0} \frac{\beta_t}{\beta_d} = 1 + 3D\Sigma_a.$$

At the other extreme of the frequency range only one finite constant is obtained:

$$\lim_{\omega \rightarrow \infty} \alpha_t = \sqrt{\frac{\Sigma_a}{2D} + \frac{3}{4} \left( \frac{1}{9D^2} + \Sigma_a^2 \right)}.$$

The ratios of the constants in the limit of infinite frequency are

$$\lim_{\omega \rightarrow \infty} \frac{\alpha_t}{\alpha_d} = 0$$

$$\lim_{\omega \rightarrow \infty} \frac{\beta_t}{\beta_d} = \infty.$$

The range of frequency-dependent curves for the attenuation and phase constants for the common moderators Be, BeO, graphite, D<sub>2</sub>O, and H<sub>2</sub>O will fall between those of H<sub>2</sub>O and D<sub>2</sub>O, the most and least absorptive of these media.

The curves for the two media are given in Fig. 4 from which a comparison is readily made. It appears that  $\alpha_d$  is greater than  $\alpha_t$  except when  $\omega = 0$  and that  $\beta_t$  is greater than  $\beta_d$ . The mathematical requirement for this is

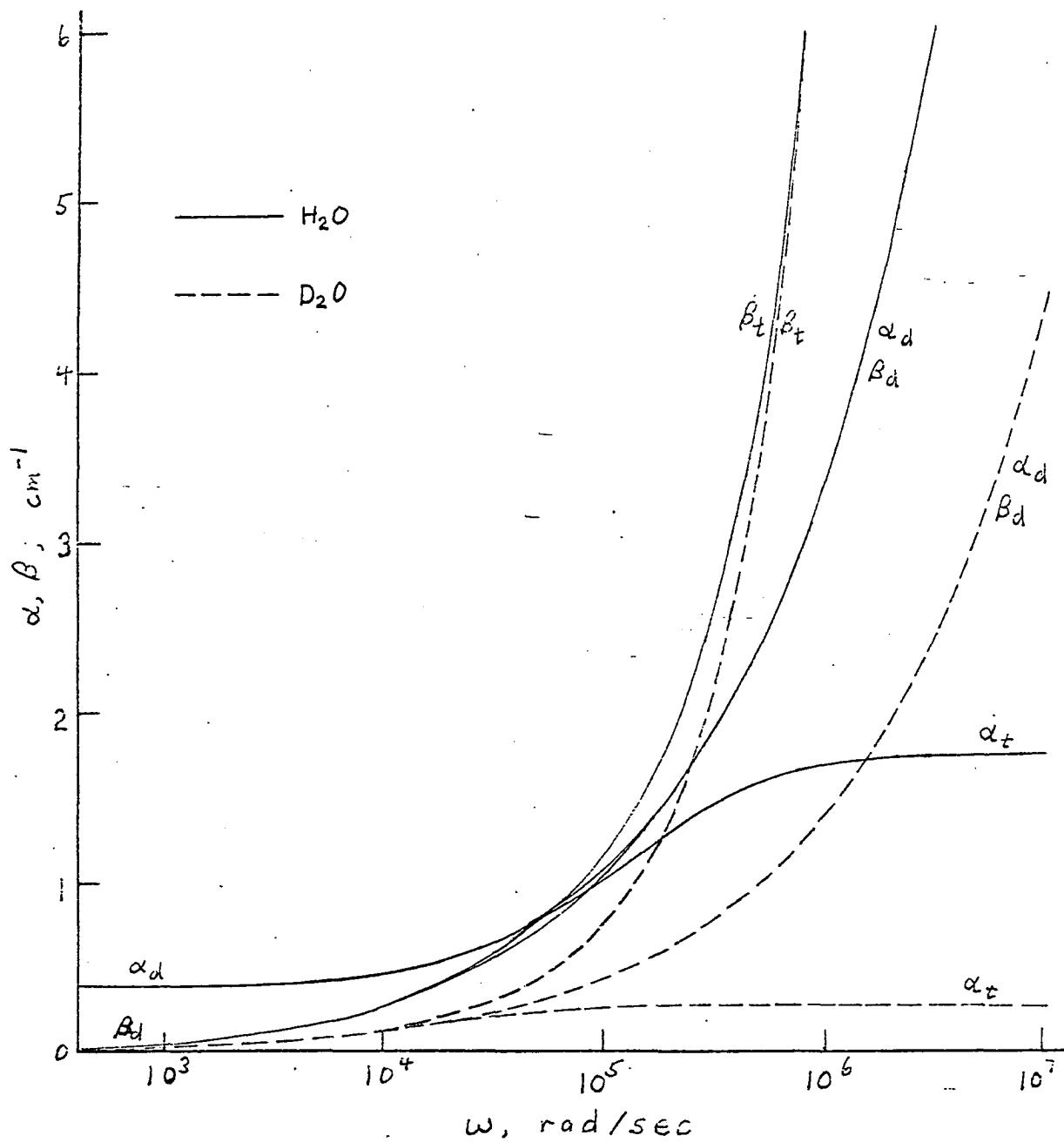


Fig. 4. Comparison of attenuation and phase constants

L<sub>4</sub>C

$$\frac{\alpha_d}{\alpha_t} > 1, \quad \frac{\beta_t}{\beta_d} > 1 \quad \left\{ \begin{array}{l} \omega > v \Sigma_a \sqrt{\left(\frac{3D\Sigma_a}{2}\right)^2 - 1} \\ \omega > 0 \end{array} \right.$$

For the materials named  $D\Sigma_a < 2/3$ .

$$D\Sigma_a > \frac{2}{3}$$

$$D\Sigma_a \leq \frac{2}{3}$$

## V. PULSED NEUTRON EXPERIMENT

In the early stages of designing a nuclear reactor many experiments are performed to measure various reactor parameters, for example, the diffusion coefficient, the diffusion length, and the mean neutron lifetime. One method for measuring some of these parameters is to pulse a test assembly with a series of short bursts of neutrons directed along a central axis normal to one face of the assembly and observe the asymptotic decay of these pulses.

An infinite slab of thickness  $h$  with a source at  $x=0$  consisting of a short burst of neutrons will be examined. A solution for this problem in the limit of an impulse burst form is derived in chapter III; however, that solution contains all the transients and reflection terms which make this form of solution too cumbersome for the experimenter. A form of the solution which will deal only with the asymptotic portion of the decay curve will now be derived.

The burst of neutrons may be either thermal or fast. For the former the burst is assumed to give rise to a flux distribution  $\Psi(x)$  after the end of the burst. For the latter the fast neutrons are assumed to be thermalized and to give rise to a thermal flux distribution  $\Psi(x)$ . (The shape of  $\Psi(x)$  will be similar but not identical for the two cases.) In the one-velocity model only the fundamental energy mode, the Maxwellian distribution is assumed to be present. To minimize the effects of the higher energy modes, the slowing-down time must be much shorter than the diffusion time so that one measures essentially the decay of the space modes only. Experimentally this condition is met by ignoring an initial part of the decay curve. For multiplying assemblies fast fission and delayed neutrons are

neglected. With short bursts and rapid pulsing the delayed neutrons appear as a constant background which is subtracted from the experimental data. The fast fission factor is approximately unity for dilute thermal systems.

One starts with the telegrapher's equation

$$\begin{aligned} \frac{\tau_0}{v} \frac{\partial^2}{\partial t^2} \phi(x, t) + \left( \frac{1}{v} + \tau_0 \sum_a \right) \frac{\partial}{\partial t} \phi(x, t) \\ = D \frac{\partial^2}{\partial x^2} \phi(x, t) - \sum_a \phi(x, t) + S(x, t) + \tau_0 \frac{\partial}{\partial t} S(x, t) \end{aligned} \quad (19)$$

and introduces the source

$$S(t) = \nu_p \sum_f p q \phi(x, t) \quad (171)$$

where  $\nu_p$  is the number of prompt neutrons per fission,  $p$  is the resonance escape probability, and  $q$  is the fast non-escape probability. Dividing equation 19 by  $D$ , substituting 171 for the source, defining the buckling

$$B^2 = \frac{1}{D} (\nu_p \sum_f p q - \sum_a) \quad (172)$$

and replacing  $vD$  by the Fick's law type diffusion coefficient  $D_0$  puts the telegrapher's equation into the form

$$\frac{\tau_0}{D_0} \frac{\partial^2}{\partial t^2} \phi(x, t) + \left( \frac{1}{D_0} + \tau_0 B^2 \right) \frac{\partial}{\partial t} \phi(x, t) = \frac{\partial^2}{\partial x^2} \phi(x, t) + B^2 \phi(x, t). \quad (173)$$

This differs from the diffusion equivalent

$$\frac{1}{D_0} \frac{\partial}{\partial t} \phi(x, t) = \frac{\partial^2}{\partial x^2} \phi(x, t) + B^2 \phi(x, t) \quad (174)$$

by the two terms involving the transport time.

To solve equation 173 one uses the separation of variables, letting

$$\phi(x, t) = \phi(x) T(t). \quad (175)$$

Dividing equation 19 by the flux and applying equation 175 produces

$$\begin{aligned} \frac{I_0}{D_0} \frac{1}{T(t)} \frac{d^2}{dt^2} T(t) + \left( \frac{1}{D_0} - \tau_0 B^2 \right) \frac{1}{T(t)} \frac{d}{dt} T(t) \\ = \frac{1}{\phi(x)} \frac{d^2}{dx^2} \phi(x) + B^2 = -\rho^2, \end{aligned} \quad (176)$$

accomplishing the separation of variables.

The spatial part is the wave equation

$$\frac{d^2}{dx^2} \phi(x) + (\rho^2 + B^2) \phi(x) = 0. \quad (177)$$

The solution for an infinite slab is

$$\phi(x) = A \cos \alpha x + C \sin \alpha x \quad (178)$$

where

$$\alpha^2 = \rho^2 + B^2. \quad (179)$$

The boundary conditions for a slab of thickness  $h$  are

$$\phi(0) = \phi(h) = 0. \quad (180)$$

The condition at the origin forces  $A$  to zero. From the other condition  $\alpha$  is replaced by  $\alpha_n$  to give

$$\alpha_n = \frac{(n+1)\pi}{h} \quad n = 0, 1, 2, \dots \quad (181)$$

Now equation 179 must be written as

$$\alpha_n^2 = \rho_n^2 + B^2. \quad (182)$$

The temporal part is

$$\frac{d^2}{dt^2} T(t) + \left(\frac{1}{\tau_0} - D_0 B^2\right) \frac{d}{dt} T(t) + \frac{D_0 \rho_n^2}{\tau_0} T(t) = 0 \quad (183)$$

Seeking a solution of the form  $e^{mt}$ , one finds the characteristic equation

$$m = -\frac{1}{2\tau_0} (1 - \tau_0 D_0 B^2) \pm \frac{1}{2\tau_0} (1 + \tau_0 D_0 B^2) \left[ 1 - \frac{4\tau_0 D_0 \alpha_n^2}{(1 + \tau_0 D_0 B^2)^2} \right]^{\frac{1}{2}} \quad (184)$$

With the Maclaurin's series representation of the square root the two values of  $m$  are

$$m_1 = D_0 B^2 - \frac{D_0 \alpha_n^2}{1 + \tau_0 D_0 B^2} - \frac{\tau_0 D_0^2 \alpha_n^4}{(1 + \tau_0 D_0 B^2)^3} - \frac{2\tau_0^2 D_0^3 \alpha_n^6}{(1 + \tau_0 D_0 B^2)^5} - \frac{5\tau_0^3 D_0^4 \alpha_n^8}{(1 + \tau_0 D_0 B^2)^7} - \dots \quad (185)$$

$$m_2 = -\frac{1}{\tau_0} + \frac{D_0 \alpha_n^2}{1 + \tau_0 D_0 B^2} + \frac{\tau_0 D_0^2 \alpha_n^4}{(1 + \tau_0 D_0 B^2)^3} + \frac{2\tau_0^2 D_0^3 \alpha_n^6}{(1 + \tau_0 D_0 B^2)^5} + \frac{5\tau_0^3 D_0^4 \alpha_n^8}{(1 + \tau_0 D_0 B^2)^7} + \dots \quad (186)$$

(For  $\tau_0 \rightarrow 0$ ,  $m_2 \rightarrow -\infty$  and  $m_1 \rightarrow -D_0 \rho_n^2$ , which is exactly the result obtained from diffusion theory.) Since the results are general, the coefficient of the  $m_2$  exponential must be zero in order that  $\phi(x, \infty) = 0$  for every  $\rho_n$ .

Therefore the initial distribution decays according to

$$\phi(x, t) = \sum_{n=0}^{\infty} A_n e^{-\lambda_n t} \sin \alpha_n x \quad (187)$$

where  $\lambda_n$  replaces  $-m_1$ . If  $\lambda_n$  is written in the form

$$\lambda_n = -D_0 B^2 + \frac{D_0 \alpha_n^2}{1 + \tau_0 D_0 B^2} \left\{ 1 + \frac{\tau_0 D_0 \alpha_n^2}{(1 + \tau_0 D_0 B^2)^2} + 2 \left[ \frac{\tau_0 D_0 \alpha_n^2}{(1 + \tau_0 D_0 B^2)^2} \right]^2 + 5 \left[ \frac{\tau_0 D_0 \alpha_n^2}{(1 + \tau_0 D_0 B^2)^2} \right]^3 + \dots \right\} \quad (188)$$

then

$$\frac{1}{1 + \tau_0 D_0 B^2} \left\{ 1 + \frac{\tau_0 D_0 \alpha_n^4}{(1 + \tau_0 D_0 B^2)^2} + 2 \left[ \frac{\tau_0 D_0 \alpha_n^4}{(1 + \tau_0 D_0 B^2)^2} \right]^2 + 5 \left[ \frac{\tau_0 D_0 \alpha_n^4}{(1 + \tau_0 D_0 B^2)^2} \right]^3 + \dots \right\}$$

is termed the telegrapher correction to the diffusion form of equation 188.

The pulsed neutron experiment is used most often in measuring neutron properties of moderators. In a non-multiplying medium the buckling  $B^2$  is simply  $-\sum_a/D$ . Recalling that the neutron lifetime  $\lambda_0$  is  $1/\nu \sum_a$ ,  $D_0 B^2$  may be replaced by  $-\lambda_0^{-1}$ . In this case

$$\lambda_n = \frac{1}{\lambda_0} + \frac{D_0 \alpha_n^2}{1 - \tau_0 / \lambda_0} \left\{ 1 + \frac{\tau_0 D_0 \alpha_n^2}{(1 - \tau_0 / \lambda_0)^2} + 2 \left[ \frac{\tau_0 D_0 \alpha_n^2}{(1 - \tau_0 / \lambda_0)^2} \right]^2 + 5 \left[ \frac{\tau_0 D_0 \alpha_n^2}{(1 - \tau_0 / \lambda_0)^2} \right]^3 + \dots \right\}. \quad (189)$$

For most moderators the transport time is much less than the lifetime. For example, the ratio  $\tau_0 / \lambda_0$  is 0.0098 for water and 0.0001 for heavy water; hence, the quantity  $1 - \tau_0 / \lambda_0$  can be taken as unity for most moderators with an error of only about one-tenth per cent or less. For water the error is about one per cent. Thus

$$\lambda_n = \frac{1}{\lambda_0} + D_0 \alpha_n^2 \left[ 1 + \tau_0 D_0 \alpha_n^2 + 2(\tau_0 D_0 \alpha_n^2)^2 + 5(\tau_0 D_0 \alpha_n^2)^3 + \dots \right] \quad (190)$$

and the telegrapher correction is the series in brackets. In the pulsed neutron experiment the fundamental mode  $\lambda_0$  is sought; in this mode  $\alpha_n^2$  is the geometric buckling  $B_g^2$ . Usually two or three terms of the correction suffice to explain the experimental results; hence, the experimenter is most concerned with the equation

$$\lambda_0 = \frac{1}{\lambda_0} + D_0 B_g^2 [ 1 + \tau_0 D_0 B_g^2 + 2(\tau_0 D_0 B_g^2)^2 ] \quad (191)$$

Equation 191 may be derived more directly by using the  $P_1$  equations, as

given by equations 13 and 14. The asymptotic time decay of the flux is assumed, so that  $\phi_n(x,t) = \phi_n(x)e^{-\lambda_n t}$ . The spatial dependence is assumed to follow  $\phi_n(x) = A_n e^{i\alpha_n x}$ . Here the subscript  $n$  refers to the  $P_n$  equations, not the  $n$ th mode, for only the fundamental mode is under consideration. Upon substituting these expressions into 13 and 14, one finds that the equations have nontrivial solutions only if

$$\begin{vmatrix} \frac{\lambda}{v} - (\sum_t - \sum_s \eta_s) & -ia \\ -ia & \frac{3\lambda}{v} - 3(\sum_t - \sum_s \eta_s) \end{vmatrix} = 0 \quad (192)$$

where  $\eta_0 = 1$  and  $\eta_1 = \bar{\mu}_0$ . Here the transport cross section is defined as  $\Sigma_{tr} = \sum_s (1 - \bar{\mu}_s)$ , which differs from the definition used previously (cf. equation 16). Condition 192 leads to 191 directly, without going through the intermediate step given by equation 189. The same result is obtained because both the assumption  $\gamma_0 \ll l$ , and the above definition of the transport cross section are based on the neglect of the absorption cross section in comparison with the scattering cross section. The importance of the telegrapher correction is illustrated in Fig. 5. Using diffusion theory one predicts a straight line relationship between  $\lambda$  and  $a^2$ , experimenters find a curve which deviates from the straight line for large  $a_0$  (small assemblies). The difference is attributed to diffusion cooling, that is, preferential leakage of faster neutrons in the smaller assemblies. Due to the telegrapher correction, or, more generally, the transport correction, the diffusion cooling effect is greater than what is apparent to the experimenter, as shown in Fig. 5. In Fig. 5 curve a is the plot of equation 191, curve b is the diffusion equivalent, and curve c is the result obtained by considering diffusion cooling and is the curve found experi-

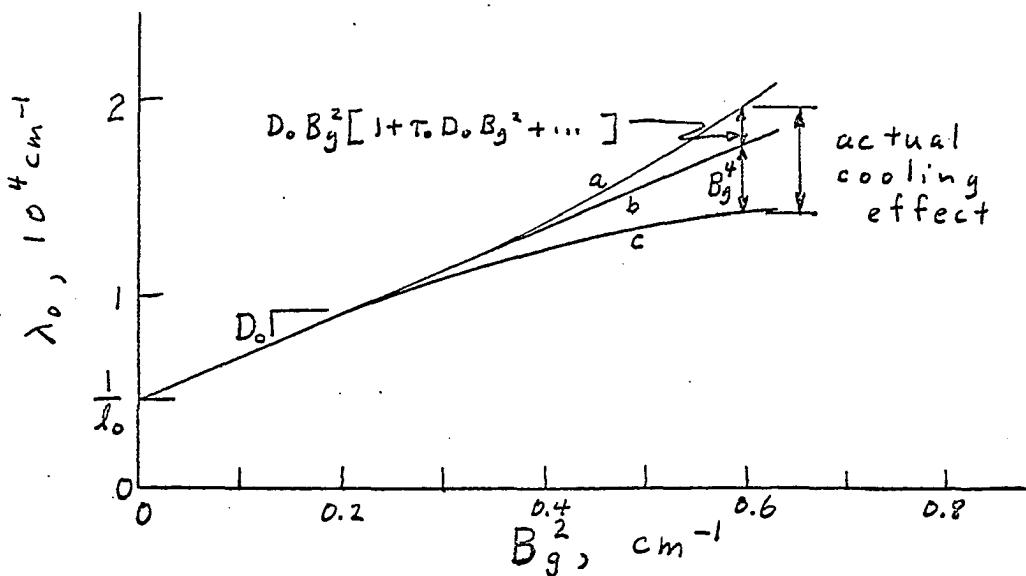


Fig. 5. Schematic diagram of decay constant vs. geometric buckling curve for a moderator. The scales are typical.

mentally. Sjöstrand (20) stated that the diffusion cooling coefficient, that is,  $C$ , should be five to thirty per cent greater than the measured value, which is found by fitting the parabola in  $B_g^2$

$$\lambda_0 = \frac{1}{\lambda_0} + D_0 B_g^2 - C B_g^4 \quad (193)$$

to the data. However, Sjöstrand found the transport correction by solving the transport equation, obtaining the condition

$$\tan \frac{B_g}{\sum_s} = \frac{B_g}{\sum_s + \sum_a - \lambda_0 / \nu} \quad (194)$$

which gives the fundamental decay constant as

$$\lambda_0 = \frac{1}{\lambda_0} + D_0 B_g^2 \left[ 1 + \frac{1}{5} \tau_0 D_0 B_g^2 + \frac{2}{35} (\tau_0 D_0 B_g^2)^2 + \dots \right] \quad (195)$$

(cf. equation 191). Nelkin (21), considering both the energy and transport corrections, is in agreement with Sjöstrand.

That the transport correction and diffusion cooling are corrections to diffusion theory of the opposite sign is also shown by Uhrig (11) in a technique closely related to neutron pulsing--sinusoidal modulation or neutron wave propagation. His plots of the real and imaginary parts of  $k(i\omega)$  for the telegrapher's equation and thermalization theory deviate in opposite directions from the straight line of diffusion theory.

The condition for the convergence of the series expansion of the square root in equation 184 will now be examined. This condition is

$$\frac{4 \tau_0 D_0 \alpha_n^2}{(1 + \tau_0 D_0 B^2)^2} < 1 \quad (196)$$

which is simplified to

$$\alpha_n < \frac{1 - 3 D \Sigma_a}{2 \sqrt{3} D} \quad (197)$$

for nonmultiplying media. It has been noted that  $1 - 3 D \Sigma_a \approx 1$  for the common moderators. If  $\alpha_0$  is replaced by  $\pi/h$  ( $h$  is the slab thickness) for the fundamental mode and  $D$  by  $1/3 \Sigma_{tr}$  or  $\lambda_{tr}/3$  ( $\lambda_{tr}$  is the transport mean free path), then the conclusion is reached that the thickness of the slab has a lower bound  $h^*$  on the order of a mean free path for which a distinct asymptotic decay constant can be found; the limiting condition is

$$h > h^* = 3.6 \sigma \lambda_{tr}. \quad (198)$$

This is in agreement with Corngold's report (22) that the critical diameter for the existence of discrete decay constants for the transport equation is of the order of a transport mean free path. Table 2 lists the limiting slab thicknesses for various moderators.

Table 2. Critical slab thicknesses for discrete decay constants

Material	$\lambda_{tr}$ , cm (23)	$h^*$ , cm
H <sub>2</sub> O	0.48	1.72
Be	1.43	5.15
D <sub>2</sub> O (0.16% H <sub>2</sub> O)	2.65	9.51
Graphite	2.75	9.90
D <sub>2</sub> O	2.86	10.30

When  $h < h^*$ , or,  $B_g^2 > B_g^{*2}$ , the decay constants of the telegrapher's equation take on complex values. Albertoni and Montagnini (24) speculate on the existence of complex decay constants in their proof that no discrete decay constants exist for  $\lambda > \lambda^*$ . Their proof, as well as that of Bednarz (25), shows that a spectral band exists, but does not reveal the nature of the spectrum. Attempts to measure the higher decay constants have been performed by Starr et al. (26) and by Möller and Sjöstrand (27) but have been only partly successful. Corngold (22) states the reason for this is that one cannot always be sure that  $\lambda_1(B_g^2)$  is seen rather than  $\lambda_0(B_g^2)$ . This problem still needs further investigation.

## VI. POINT REACTOR KINETICS

## A. The Kinetics Equations

In a nuclear reactor containing fissionable material the delayed neutrons must be considered. A classic approach to reactor kinetics is to describe the reactor by the neutron flux and delayed neutron precursor equations, integrated over volume and lethargy. This simple picture has proven very useful in the safety analysis of nuclear power plants (27) and in the analysis of experiments (28).

The classic derivation (28, 29) of the kinetics equations involved integrating the Boltzmann transport equation over space, direction, and energy and defining basic reactor parameters in terms of the resulting integrals. A more elementary derivation (30) reduces the usual monoenergetic diffusion equations to the point reactor kinetics equations. In this chapter the telegrapher's equation will be the starting point and the derivation will parallel that of Danofsky (30).

For a fission source with delayed neutrons

$$S(x, t) = (1-\beta) k q \sum_a \phi(x, t) + p q \sum_i \lambda_i c_i(x, t) \quad (199)$$

$$\frac{\partial}{\partial t} c_i(x, t) = -\lambda_i c_i(x, t) + \frac{k}{p} \beta_i \sum_a \phi(x, t). \quad (200)$$

With  $S(x, t)$  given by 199 the telegrapher's equation takes the form

$$\begin{aligned} & \frac{\tau_0}{v} \frac{\partial^2}{\partial t^2} \phi(x, t) + \left[ \frac{1}{v} + \tau_0 \sum_a - \tau (1-\beta) k q \sum_a \right] \frac{\partial}{\partial t} \phi(x, t) - \tau_0 p q \\ & \cdot \sum_i \lambda_i \frac{\partial}{\partial t} c_i(x, t) = D \frac{\partial^2}{\partial x^2} \phi(x, t) - \sum_a [1 - (1-\beta) k q] \phi(x, t) \\ & \quad + p q \sum_i \lambda_i c_i(x, t). \end{aligned} \quad (201)$$

The comparable diffusion equation is

$$\frac{1}{v} \frac{\partial}{\partial t} \phi(x, t) = D \frac{\partial^2}{\partial x^2} \phi(x, t) - \sum_a [1 - (1-B) k_g] \phi(x, t) + p g \sum_i \lambda_i c_i(x, t). \quad (202)$$

To derive the kinetics equations, separability of space and time is assumed; then

$$\phi(x, t) = \phi(x) T(t) \quad (203)$$

$$c_i(x, t) = c_i(x) H_i(t). \quad (204)$$

Equation 200 will be examined first. With equations 203 and 204

$$c_i(x) \frac{d}{dt} H_i(t) = -\lambda_i c_i(x) H_i(t) + \frac{k}{p} B_i \sum_a \phi(x) T(t). \quad (205)$$

The definitions

$$\lambda = \frac{\lambda_0}{1 + L^2 B^2} \quad (206)$$

$$k_{eff} = \frac{k_g}{1 + L^2 B^2} \quad (207)$$

of the finite mean neutron lifetime and effective multiplication factor, the definition  $\phi(x) = v n(x)$ , and the assumption that  $p g \approx 1$  simplify 205 to

$$c_i(x) \frac{d}{dt} H_i(t) = -\lambda_i c_i(x) H_i(t) + \frac{B_i k_{eff} n(x) T(t)}{\lambda}. \quad (208)$$

Integrating over the reactor volume, letting

$$\bar{c}_i = \frac{1}{V} \int_V c_i(x) dV \quad \bar{n} = \frac{1}{V} \int_V n(x) dV \quad (209)$$

and calling

$$\bar{C}_i H_i(t) = c_i(t) \quad (210)$$

$$\bar{n} T(t) = n(t) \quad (211)$$

puts equation 208 into the form

$$\frac{d}{dt} C_i(t) = -\lambda_i C_i(t) + \frac{\beta_i k_{eff}}{\ell} n(t). \quad (212)$$

Equation 201 will now be put into the same form as 212. Using equations 203 and 204,  $\phi(x,t)$  and  $C_i(x,t)$  are again separated in space and time.

From the first fundamental theorem of reactor theory

$$\frac{d^2}{dx^2} \phi(x) = -B^2 \phi(x). \quad (213)$$

The definitions of flux, finite neutron lifetime, and effective multiplication factor and the assumption that  $p \approx 1$  are used again; also used is the definition of diffusion length  $L = \sqrt{D/\Sigma_a}$ . Now the telegrapher's equation may be written as

$$\begin{aligned} \tau_0 n(x) \frac{d^2}{dt^2} T(t) + \left[ 1 + \frac{\tau_0}{L} - \frac{\tau_0 (1-\beta) k_{eff}}{L} \right] n(x) \frac{d}{dt} T(t) - \tau_0 \sum_i \lambda_i C_i(x) \frac{d}{dt} H_i(t) \\ = -\frac{n(x) T(t)}{L} - \frac{(1-\beta) k_{eff}}{L} n(x) T(t) + \sum_i \lambda_i C_i(x) H_i(t). \end{aligned} \quad (214)$$

Integrating over the reactor volume and using equations 210, 211, and 212 changes equation 214 to

$$\begin{aligned} \tau_0 \frac{d^2}{dt^2} n(t) + \tau_0 \left[ \frac{1}{\tau_0} + \frac{1}{L} - \frac{(1-\beta) k_{eff}}{L} \right] \frac{d}{dt} n(t) - \tau_0 \sum_i \lambda_i \frac{d}{dt} C_i(t) \\ = \frac{(1-\beta) k_{eff} - 1}{L} n(t) + \sum_i \lambda_i C_i(t). \end{aligned} \quad (215)$$

The coefficient  $[(1-\beta) k_{eff} - 1]/L$  in equation 215 is the net gain of prompt neutrons per unit time. As part of the coefficient of  $dn(t)/dt$ ,

the total gain rate  $(1-\beta)k_{\text{eff}}/l$  appears with the sum of the inverses of the infinite transport time and infinite lifetime instead of with the inverse of the finite lifetime. If one adds and subtracts the inverse finite lifetime in the brackets in front of  $d\psi(t)/dt$ , he gets the net rate of gain of prompt neutrons and the quantity

$$\frac{1}{\tau_0} + \frac{1}{\lambda_0} - \frac{1}{\lambda} = \frac{\nu}{3D} (1 - 3D^2 B^2). \quad (216)$$

Now the finite mean neutron transport time  $\tau$  is defined; it is

$$\tau = \frac{\tau_0}{1 - 3D^2 B^2}. \quad (217)$$

Equation 216 now becomes the identity

$$\frac{1}{\tau} - \frac{1}{\tau_0} + \frac{1}{\lambda} - \frac{1}{\lambda_0} = 0. \quad (218)$$

Although  $\tau$  and  $\lambda$  are functions of the buckling  $B^2$  and vary with it in the opposite sense, the identity clearly is independent of  $B^2$ ; hence, 218 states that a decrease in the lifetime (by increasing  $B^2$ ) is compensated by an increase in the transport time, in a reciprocal fashion.

While the lifetime increases due to the higher leakage, the transport time decreases due to the larger gradient of the flux. Since the normalized flux shape is "bent" more for a smaller system, the gradient of the flux is greater and the diffusion is increased. Three mechanisms are possible:

- i. an increase in the neutron velocity,
- ii. an increase in the transport mean free path, and
- iii. an increase in the transport time.

For the monoenergetic model the first two parameters are constants; there-

fore the transport time must be increased. The finite transport time is seen as the effective time lapse over which the transport process takes place for a neutron traveling at velocity  $v$  in a material with transport mean free path  $\lambda_{tr}$ .

The kinetics equations may now be written

$$\begin{aligned} \tau_0 \frac{d^2 n}{dt^2} + \tau_0 \left[ \frac{1}{\tau} - \frac{(1-\beta) k_{eff} - 1}{\ell} \right] \frac{dn}{dt} - \tau_0 \sum_i \lambda_i \frac{dc_i}{dt} &= \\ = \frac{(1-\beta) k_{eff} - 1}{\ell} n + \sum_i \lambda_i c_i & \end{aligned} \quad (219)$$

$$\frac{dc_i}{dt} = -\lambda_i c_i + \frac{\beta_i k_{eff}}{\ell} n \quad (220)$$

where the arguments of  $n$  and  $c_i$  have been dropped. Compare this with the set derived from diffusion theory

$$\frac{dn}{dt} = \frac{(1-\beta) k_{eff} - 1}{\ell} n + \sum_i \lambda_i c_i \quad (221)$$

$$\frac{dc_i}{dt} = -\lambda_i c_i + \frac{\beta_i k_{eff}}{\ell} n \quad (222)$$

The time dependent portion of the neutron balance equation has undergone gross changes; the remainder is identical to the diffusion result.

### B. Reactor Transfer Function

A popular and useful technique for frequency analysis of a reactor close to the critical state is to determine the reactor transfer function for a linearized reactor model. By assuming a reactor close to critical one can take  $k_{eff} \approx 1$ . The excess multiplication  $k_{ex} = k_{eff} - 1$  is represented by  $\delta k$  since only small variations from the steady state are considered. Now one may write  $(1-\beta) k_{eff} - 1 = \delta k - \beta$ . For small changes in  $n$  and  $c_i$  due

to  $\delta k$  the functions  $n$  and  $C_i$  may take the form

$$n = n(0) + \delta n \quad (223)$$

$$C_i = C_i(0) + \delta C_i \quad (224)$$

where  $\delta n$ ,  $\delta C_i$ , and  $\delta k$  are functions of time. Equations 219 and 220 now become

$$\begin{aligned} \tau_0 \frac{d^2}{dt^2} \delta n + \tau_0 \left[ \frac{1}{\tau} - \frac{\delta k - \beta}{\ell} \right] \frac{d}{dt} \delta n - \tau_0 \sum_i \lambda_i \frac{d}{dt} \delta C_i \\ = \frac{\delta k - \beta}{\ell} [n(0) + \delta n] + \sum_i \lambda_i [C_i(0) + \delta C_i] \end{aligned} \quad (225)$$

$$\frac{d}{dt} \delta C_i = -\lambda_i [C_i(0) + \delta C_i] + \frac{\beta_i}{\ell} [n(0) + \delta n]. \quad (226)$$

At steady state  $\lambda_i C_i(0) = \beta_i n(0)/\ell$  and  $\delta k(0) = 0$ . Under these conditions equations 225 and 226 are simplified to

$$\begin{aligned} \tau_0 \frac{d^2}{dt^2} \delta n + \tau_0 \left[ \frac{1}{\tau} - \frac{\delta k - \beta}{\ell} \right] \frac{d}{dt} \delta n - \tau_0 \sum_i \lambda_i \frac{d}{dt} \delta C_i \\ = \frac{n(0)}{\ell} \delta k - \frac{\beta}{\ell} \delta n + \sum_i \lambda_i \delta C_i \end{aligned} \quad (227)$$

$$\frac{d}{dt} \delta C_i = -\lambda_i \delta C_i + \frac{\beta_i}{\ell} \delta n. \quad (228)$$

Neglecting the products  $\delta k \delta n$  and  $\delta k \frac{d \delta n}{dt}$  in these equations will effectively linearize the equations. One may ask whether the second product can be neglected at high frequencies. It cannot at very high frequencies where  $\omega > \frac{1}{\tau_0}$ , which is on the order of  $10^5$  rad/sec for most moderators, but such frequencies are much higher than those of interest to this problem. The next procedure is to find the reactor transfer function.

Laplace transforming equations 227 and 228 results in

$$\begin{aligned}\tau_0 s^2 \Delta N + \tau_0 \left[ \frac{1}{\tau} + \frac{\beta}{\lambda} \right] s \Delta N - \tau_0 \sum_i \lambda_i s \Delta C_i \\ = \frac{n(\sigma)}{\lambda} \Delta K - \frac{\beta}{\lambda} \Delta N + \sum_i \lambda_i \Delta C_i\end{aligned}\quad (229)$$

$$s \Delta C_i = -\lambda_i \Delta C_i + \frac{\beta_i}{\lambda} \Delta N \quad (230)$$

where  $\Delta N$ ,  $\Delta C_i$ , and  $\Delta K$  are the Laplace transforms of  $\delta n$ ,  $\delta C_i$  and  $\delta K$ , respectively, and are functions of  $s$ . The second equation is

$$\Delta C_i = \frac{\beta_i / \lambda}{s + \lambda_i} \Delta N. \quad (231)$$

Eliminating  $\Delta C_i$  in equation 229, using equation 231, and solving for  $\frac{\Delta N}{\Delta K}$

gives

$$\frac{\Delta N}{\Delta K} = \frac{\frac{n(\sigma)}{\lambda \tau_0}}{s \left[ s + \frac{1}{\tau} + \frac{\beta}{\lambda} \right] - \sum_i \frac{(\lambda_i - \frac{1}{\tau_0}) \frac{\beta_i}{\lambda}}{s + \lambda_i}}. \quad (232)$$

This ratio is the transfer function of the input  $\Delta K$  and the output  $\Delta N$  for a reactor with delayed neutrons at zero power, that is, without consideration of temperature and fluid flow effects found in reactors operating at non-zero power levels.

An important simplification of the transfer function can be produced by considering the case of one delayed group of neutrons. If the frequency range of interest is not extended too low, this model adequately describes the behavior of many thermal reactors. Let  $\lambda$  be the decay constant for the one group model. In this case

$$\frac{\Delta N}{\Delta K} = \frac{n(\sigma)}{\lambda \tau_0} \frac{s + \lambda}{s \left[ s^2 + (\lambda + \frac{1}{\tau} + \frac{\beta}{\lambda}) s + \frac{\lambda}{\tau} + \frac{\beta}{\tau_0 \lambda} \right]}. \quad (233)$$

For thermal reactors one may assume

$$\lambda \ll \frac{1}{\tau}$$

$$\lambda \ll \frac{\beta}{\ell}$$

When  $3D^2\beta^2 \ll 1$ , so that  $1 - 3D^2\beta^2 \approx 1$ , then  $\tau \approx \tau_0$ . Hence for one group of delayed neutrons

$$\frac{\Delta N}{\Delta K} = \frac{n(0)}{\lambda \tau} \frac{s + \lambda}{s(s + \beta/\ell)(s + 1/\tau)}. \quad (234)$$

This may be compared with the corresponding diffusion model:

$$\frac{\Delta N}{\Delta K} = \frac{n(0)}{\lambda} \frac{s + \lambda}{s(s + \beta/\ell)}. \quad (235)$$

The telegrapher's model increases the magnitude by a factor  $1/\tau$  and adds a pole at  $1/\tau$ . As in the case of the steady state response to a sinusoidal source located at a boundary,  $s$  is replaced by  $j\omega$  to determine the frequency response in the reactor, this time due to the oscillations in  $\delta k$  taking place in the interior of the reactor. The frequency responses of the two models for the one group case are compared in Fig. 6 for the magnitude and in Fig. 7 for the phase. The effect on the magnitude is to put a breakpoint at  $\omega = 1/\tau$  and increase the slope thereafter by another 20 db/decade. For the phase the effect is to shift the phase down another  $90^\circ$  for frequencies higher than  $1/\tau$ . A transition zone two decades wide and centered on  $\omega = 1/\tau$  shifts the phase down. Different lifetimes have a large effect on the shape of the transfer functions; different transport times have little effect. The curves for several lifetimes are shown in Fig. 6 and 7, but only the breakpoints for several transport times are shown, since the curves for different transport times would fall almost on top of each other.

One interesting point is that the breakpoint for the transport time is

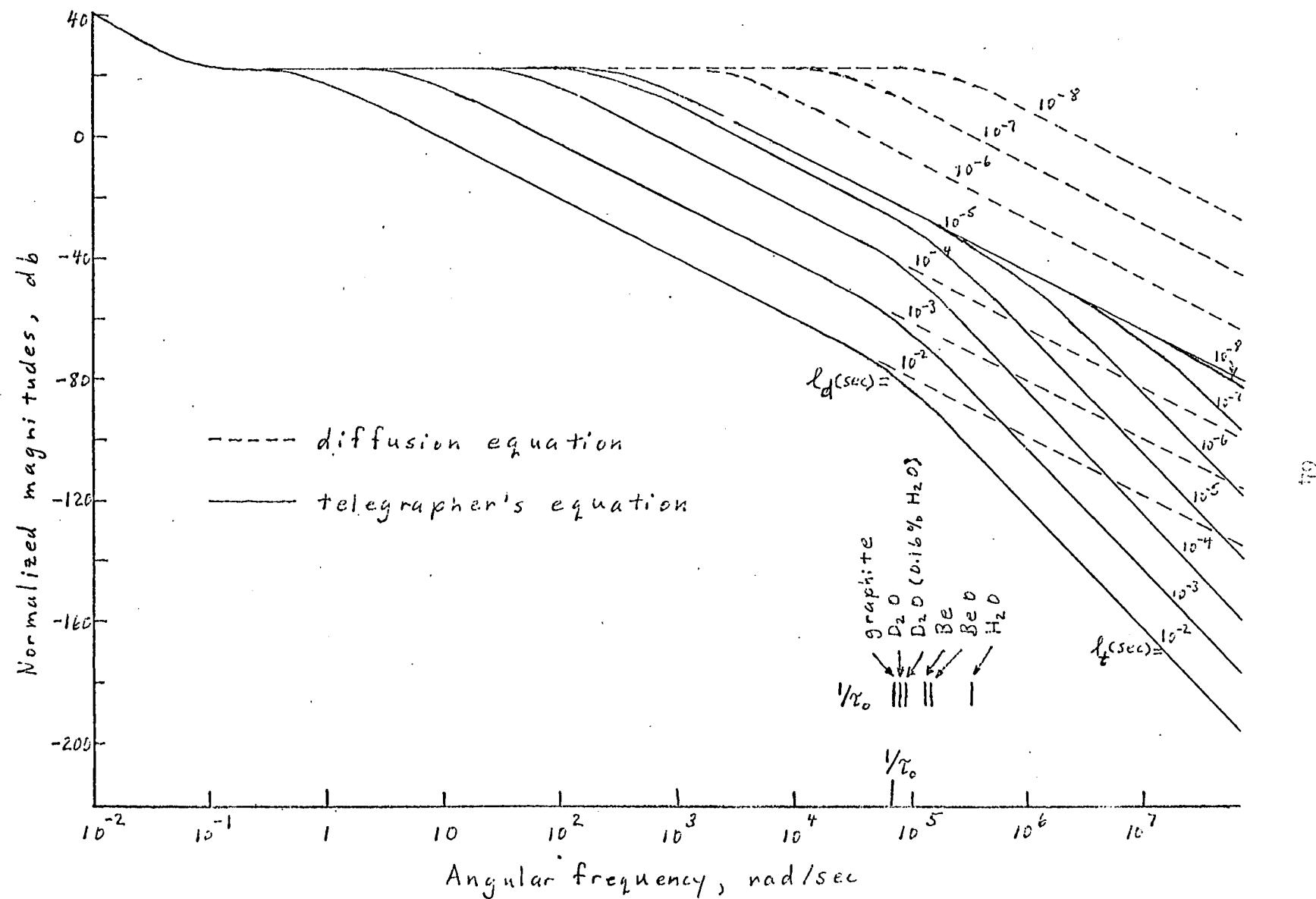


Fig. 6. Comparison of the magnitude of the transfer functions for one group of delayed neutrons (equations 233 and 235) in graphite. The magnitudes have been normalized to the plateau at 22 db.

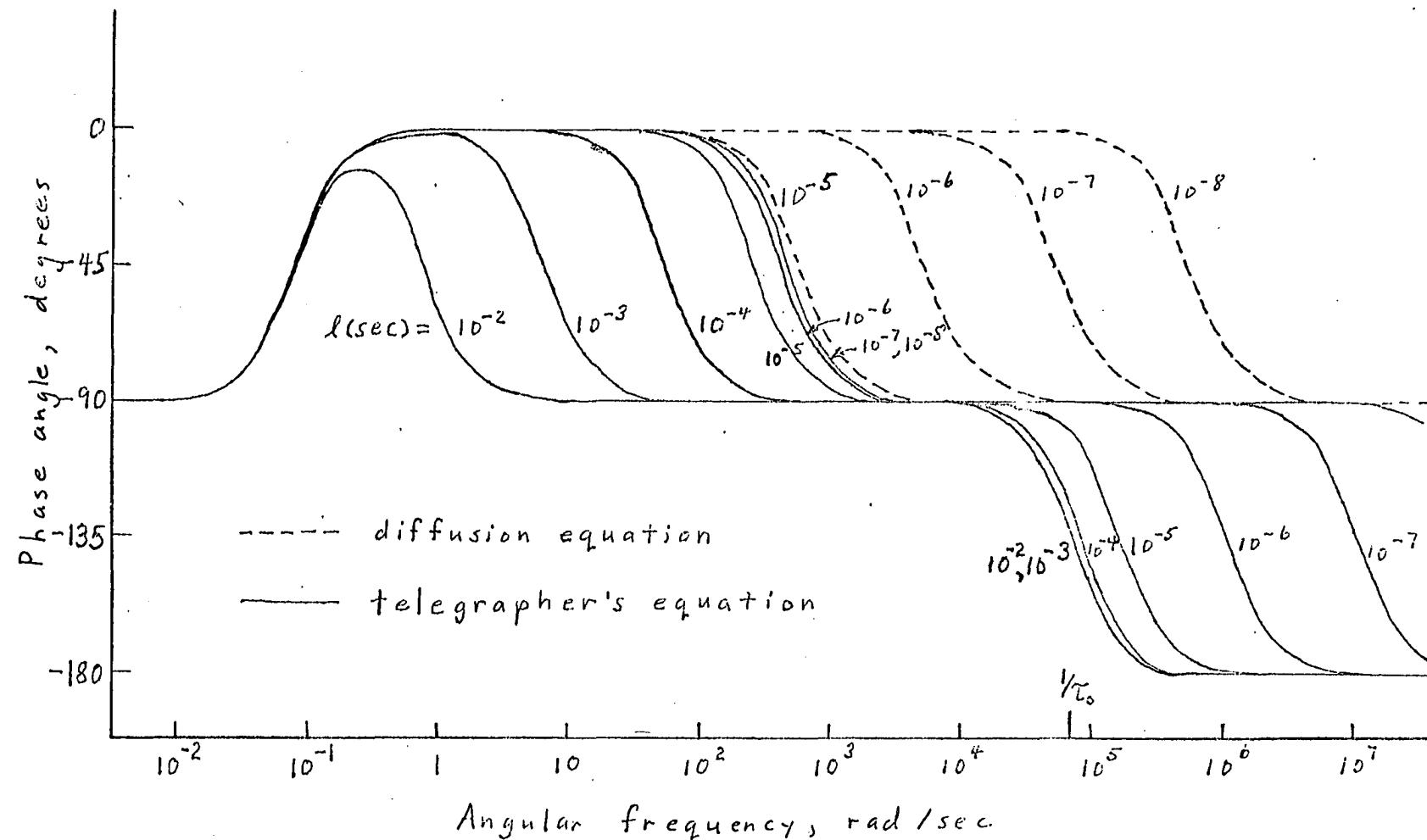


Fig. 7. Comparison of the phase of the transfer functions for one group of delayed neutrons (equations 233 and 235) in graphite.

sufficiently independent of the buckling that it provides another method for measuring the diffusion coefficient or transport cross section for the reactor. For thermal reactors the diffusion coefficient is usually taken to be that of the moderator; it is essentially the diffusion coefficient of the moderator that is determined when this breakpoint is found. The fundamental problem in an experiment to determine this upper breakpoint is to obtain an oscillator capable of such high frequencies. Usually the frequencies are less than 1000 rad/sec for transfer function measurements (31).

## VII. SUMMARY

Solutions for the telegrapher's equation were found for two types of infinite systems, the infinite slab and the semi-infinite medium, each with a source at one boundary and each having zero initial conditions. Some limitations to diffusion theory are revealed by the investigation of the asymptotic behavior of a pulse source. The asymptotic decay constant is not a straight line function of the geometric buckling as in diffusion theory, but has a small curvature, and the discrete decay constant no longer exists for systems with a dimension on the order of a transport mean free path, a result which is consistent with transport theory but is not found in diffusion theory.

The frequency response was determined for an external sinusoidal source and for an interior source oscillating uniformly throughout the reactor volume. In the latter case the reactor transfer function for one group of delayed neutrons differs from the corresponding diffusion transfer function by an additional pole at  $1/\tau_0$ , and by a magnitude increase of  $1/\tau_0$ . In both cases diffusion theory agrees with the telegrapher's equation for frequencies less than  $10^4$  rad/sec. The divergence above this is due to the presence of the mean transport time  $\tau_0$ , whose inverse is on the order of  $10^5$  rad/sec for common moderators.

The parameter  $\tau_0$  is the distinguishing feature between the diffusion and telegrapher's equations. The finite transport time  $\tau$ , which was introduced in this work, was found to be important conceptually and provided a symmetric identity with the mean neutron lifetimes  $\lambda_0$  and  $\lambda$ .

## VIII. CONCLUSIONS

The telegrapher's equation provides an important extension to diffusion theory. In the usual range of reactor calculations, the telegrapher's equation agrees with the diffusion equation, the latter being simpler to use. It was found that the telegrapher's equation is particularly useful in calculations in pulsed neutron experiments. By using this equation, the effects of the delay due to the mean neutron transport time may be examined; diffusion theory neglects these effects. The determination of limitations on diffusion theory are perhaps the most useful results obtained in this investigation. The telegrapher's equation will be useful to experimenters when the frequency range of oscillators used in transfer function measurements is extended to much higher frequencies, say by at least two decades.

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## V. ACKNOWLEDGMENTS

The author wishes to express his gratitude to Dr. Richard A. Donofsky for his interest and encouragement during this project. He also wants to thank his committee members for their constructive criticism.

The author is grateful to the Atomic Energy Commission and to the Department of Nuclear Engineering at Iowa State University for two years of support on an Atomic Energy Commission Traineeship, which enabled him to study as a full-time student.

The author's family has been a constant source of comfort and encouragement, for which he is greatly appreciative.